1. Probability: Problem of Points

   The game
   A fair coin is repeatedly flipped. If the coin lands as a heads (H), Hilda gets a point. If the coin lands tails, Tom gets a point. The first person to 5 points wins. Each player pays $10 to play.

   You will fill out the table below indicating how much each player should get assuming the game is stopped before finishing. You must justify your calculations using **both** Pascal's method and Fermat's method.

	status of game at termination	payout to Hilda	payout to Tom
a)	Hilda 4, Tom 4		
b)	Hilda 4, Tom 3		
c)	Hilda 4, Tom 2		
d)	Hilda 4, Tom 1		
e)	Hilda 4, Tom 0		
f)	Hilda 3, Tom 2		

2. In class, I described Pascal's Method as **recurrsive**. Using your work from problem 1, explain, in your own words, why this is an accurate description of this method.

3. Give an argument that Fermat's strategy is clear and correct theoretically but may be challenging to implement in specific instances.

4. (perfect numbers)

   (a) Use the definition to show that 496 is a perfect number.

   (b) Show that $496 = 2^{k-1}(2^k - 1)$ for some integer k and $2^k - 1$ is prime.

5. Euler and Number Theory
   Recall that Euclid proved: If $2^k - 1$ is prime ($k > 1$), then $n = 2^{k-1}(2^k - 1)$ is perfect.
   Euler proved that if n is an even perfect number, then $n = 2^{k-1}(2^k - 1)$ for some integer k and $2^k - 1$ is prime.
   Complete the following using the results above.

   (a) Show that $8128 = 2^6(2^7 - 1)$ is perfect.

   (b) Show that $2096128 = 2^{10}(2^{11} - 1)$ is not perfect.

   (c) We know there are an infinite number of primes. Are there an infinite number of perfect numbers?

   (d) What is the first odd perfect number?

6. Use Euler's method to show $\sum_{i=1}^{\infty} \frac{1}{2i-1} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

   Hint: Use the Maclaurin Series for $\cos(x)$ and the fact that $\cos(x) = 0$ has roots $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, ...