Spring 2025 **Math 316**

Exam 1

Name: Solutions

Rules:

- Partial credit will be awarded, but you must show your work.
- No notes, books, or cell phones are allowed.
- Calculators are allowed.

NOTE: The exam is formatted in order to provide plenty of space. A complete answer does **not** need to fill the given space. Indeed, it is not necessary.

Problem	Possible	Score
1	15	
2	15	
3	20	
4	20	
5	15	
6	15	
Extra Credit	(5)	
Total	100	

1. (15 points) Use ancient Egyptian methods to complete the following calculations.

   (a) Multiply 50 by 42.

				OR			
	1	50				1	42	
✓	2	100	←		✓	2	84	←
	4	200				4	168	
✓	8	400	←			8	336	
	16	800			✓	16	672	←
✓	32	1600	←		✓	32	1344	←
	42	2100				50	2100	

   (b) Divide 159 by 42. $\frac{159}{42} = x$ or $159 = x \cdot 42$. Find x.

1	42	✓
2	84	✓
$\frac{1}{2}$	21	✓
$\frac{1}{4}$	$10\frac{1}{2}$	✓
$\frac{1}{42}$	1	✓
$\frac{1}{84}$	$\frac{1}{2}$	✓
$3\,\frac{1}{2}\,\frac{1}{4}\,\frac{1}{42}\,\frac{1}{84}$	159	

aside
running total
• $84+42=126$
need 33
-21
need 12
-10.5
need 1.5

   (c) Show that unit fraction decomposition is not unique by giving an example of a rational number q and two different unit fraction decompositions of q (where the unit fractions must be distinct).

$\frac{7}{12} = q$

$\frac{7}{12} = \boxed{\frac{1}{2} + \frac{1}{12}} = \frac{3}{12} + \frac{4}{12} = \boxed{\frac{1}{4} + \frac{1}{3}}$

2. (15 points) This problem concerns the base-60 numerical representation of the ancient Babylonians. We will use our textbook's notation.

   (a) Write the base 60 number 3; 45 in base 10.

   $3 + \frac{45}{60} = 3.75$

   (b) Explain, with computation, how 3; 45 can be the reciprocal of 16.

   $\underbrace{16 \cdot 3;45}_{\text{base 60}} = \underbrace{16 \cdot (3.75)}_{\text{base 10}} = \underbrace{60}_{\text{base 10}} = \underbrace{1,0}_{\text{base 60}}$

   Since Babylonians had no sexegesimal point, the magnitude of the number was flexible, or taken by context. In this context the base 60 number 10 was taken as 1.

   (c) Describe two advantages of the Babylonian base 60 representation of numbers when compared to ancient Egyptian representation.

   - fewer symbols to memorize. Babylonians had only two. Ancient Egyptians had many.
   - flexible fractional notation. Babylonians used a sexegesimal system like ~~or~~ our decimal system. Egyptians insisted on distinct unit fractions.
   - compact representation of large numbers. Babylonians could represent numbers over 200,000 using only two symbols and three positions. Egyptians needed to have 6 symbols with repeats.
   - easier computation. Much less memorization to do $+, -, \cdot, \div$.

3. (20 points) This question is about the idea of **incommensurable quantities.**

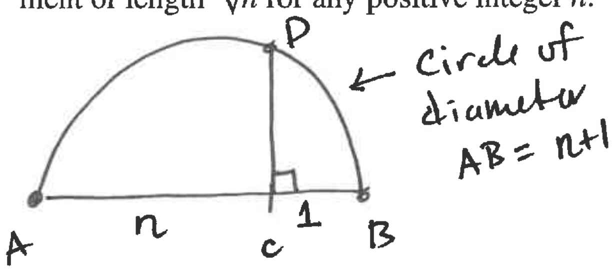
(a) What did ancient Greek mathematicians mean by the words "line segments AB and BC are incommensurable"?

There does not exist another line segment, DE, so that ~~AE~~ AB and BC are BOTH integer multiples of DE.

(b) How, when, and who discovered the existence of incommensurable line segments.

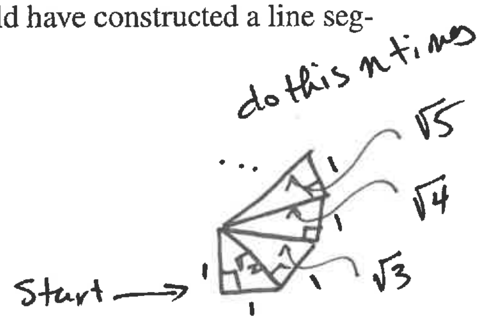
Probably the Pythagoreans around 600-500 BC. They found the sides of a square ~~were~~ and its ~~th~~ diagonal were incommensurable.

(c) Describe (with pictures) how ancient Greek mathematicians could have constructed a line segment of length $\sqrt{n}$ for any positive integer n.



CD would have length $\sqrt{n}$

or



The hypotenuse will eventually be $\sqrt{n}$.

(d) Why is the existence of incommensurable quantities a crisis for Greek mathematicians?

This creates a conflict between numbers – that is rational numbers – and lengths (or areas or volumes.) Thus, there are lengths that cannot be described as numbers.

4. (20 points) This question is about Euclid's **Elements**.

(a) When and where was it written?

Alexandria, 300 BC

(b) Describe its contents including its structure and its style of exposition.

Contents: Two and three dimensional geometry, number theory, and theory of proportions

Structure: 13 books, beginning with axioms and definitions followed by propositions and proofs

Style: Strictly logical structure with theorem followed by proof. No intuition or foreshadowing or motivation.

(c) State the 5th Postulate and explain how it is different from the other four postulates.

- If one line falls on two lines so that the sum of angles on one side is less than two right angles, then the two lines meet on that side.
- way longer and more complicated and less intuitive than the other 4.

(d) Compare Euclid's proof of the infinitude of primes from a modern one.

It has the same logical structure, namely by contradiction, as the modern one. It also makes use of the same contradictory number, namely the sum of known primes plus 1. The modern one use indices to denot an arbitrary number of primes, where Euclid used 3.

(e) State two notable results found in Euclid's **Elements** (other than the proposition that there are an infinite number of primes.)

He proved the Pythagorean Theorem and its converse.

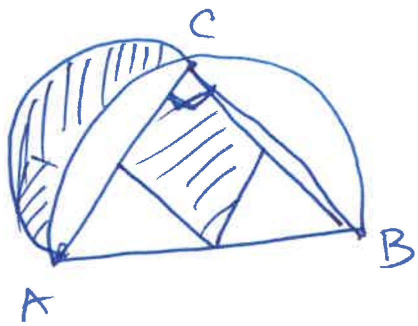
There are so many answers here such as

- Sums of geometric sequences
- Construction of 5 regular Platonic solids
- formula for Pythagorean triples.
- Euclidean algorthm for finding gcd(a,b)

5. (15 points) This question is concerns the notion of the quadrature of geometric figures.

(a) Explain the what is meant by the statement that Hippocrates accomplished the quadrature of a lune and explain why this was important. (You may want to draw pictures to illustrate your words.)

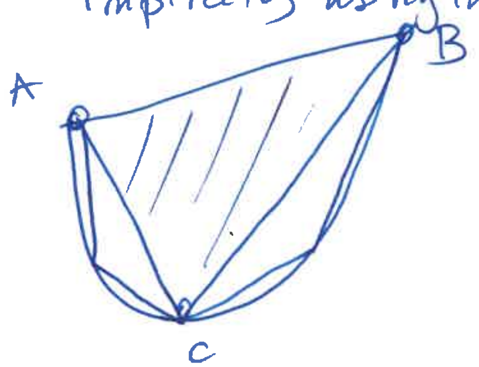
He showed that the lune (shaded) had the same area as the square (shaded) ~~too~~ where ΔABC is an isosceles right triangle.



He hoped this was a first step toward squaring the circle.

(b) Describe the method Archimedes used in his quadrature of a parabolic segment. (You may want to draw pictures to illustrate your words.)

He showed that the area of the parabolic segment was $\frac{4}{3}$ the area of a particular triangle (ΔABC). He did this by summing a series of triangular regions filling the parabolic segment. While Archimedes uses ~~the~~ the method of proof by contradiction, he is implicitly using the notion of a limit.



6. (15 points) Short answer.

(a) Explain how Archimedes' statement that **the area of a circle is to the square on its diameter as 11 is to 14** can be interpreted as an estimation of π.

$A : d^2 = 11:14$ or $\frac{A}{d^2} = \frac{11}{14}$ or $A = \frac{11}{14} d^2$.

For us, $A = \pi r^2 = \frac{\pi}{4} d^2$. So, we conclude

$$\frac{\pi}{4} \approx \frac{11}{14} \quad \text{or} \quad \pi \approx \frac{44}{14} = 3.142857\ldots$$

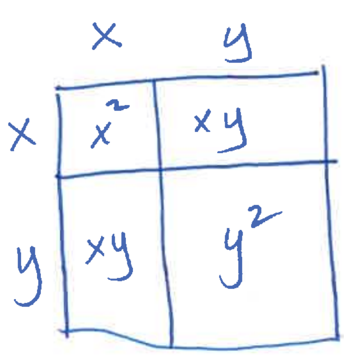
(b) Describe two ways in which Diophantus' notation in **Arithmetica** was novel and describe two of its limitations.

novel : He used symbols for x, x^2, x^3 and for subtraction and equality.

limitations : · No parantheses meant the notation required careful arrangement. No symbols for $+$ or $\times$ or $\div$.
All symbols for x, x^2, x^3 were different

(c) Give one example of cut-and-paste algebra. A complete answer includes a picture (or pictures) and a modern algebraic identity that the picture (or pictures) demonstrates.

$(x+y)^2 = x^2 + 2xy + y^2$



The whole area is $(x+y)^2$.

It's split into pieces of the form x^2, $2xy$, y^2

Extra Credit: (5 points) Below is Problem 27 from Book I from Diophantus' **Arithemetica** as it appears in Thomas Heath's 1910 translation. Provide a modern algebraic explanation of the problem and the solution. Full points for explaining the trick Diophantus is employing that make his scheme work.

To find two numbers such that their sum and product are given numbers.

goal: Find x, y so that $x+y=a$ and $xy=b$ where a and b are given (and $x, y \in \mathbb{Q}^+$)

Given sum 20, given product 96.
2x the difference of the required numbers.
Therefore the numbers are $10 + x$, $10 - x$.
Hence, $100 - x^2 = 96$.
Therefore, $x = 2$, and
the required numbers are 12, 8.

translation

Suppose $a=20$ and $b=96$

* $x-y=2z$ (Dang! I had to replace x with z because I used x, y earlier and I wrote in pen...)

So $x=10+z$ and $y=10-z$.

So $xy=(10+z)(10-z)=100-z^2=96$

So $4=z^2$ and $z=2$.

So $x=10+2=12$ and $y=10-2=8$.

* Trick: Use is one unknown to be half the difference!
This forces xy to ALWAYS be of the form..
$\left(\frac{a}{2}\right)^2 - z^2 = b$ So it's easy to solve for z.