Solutions

Part I

This part is written without notes or aids of any kind. It is worth 25 points out of 100 total points.

Below is a list of eight mathematicians or mathematical documents. Give a detailed description of **five of the eight**. A complete description will include approximate dates, location, and mathematical significance.

Once you have completed Part I and turn it in, you will be given Part II. You cannot return to Part I once it has been turned in.

1. Rhind Papyrus
   (1650 bc, Egypt) It contains a series of problems or exercises such as how to divide loaves of bread between several men. Many of the problems are solved using the method of false position. At its beginning, it contains a table showing how to write fractions of the form 2/n as sums of unit fractions.

2. Moscow Papyrus
   (1850 bc, Egypt) It contains a series of problems or exercises. In one problem, a formula volume of a truncated pyramid is demonstrated and it is the correct one. In another example, the author appears to give the correct formula for the surface area of a hemisphere, using a reasonable approximation of π.

3. Plimpton 322
   (1900-1700 bc, Babylon) It is an old Babylonian tablet on which are written a list of 15 primitive Pythagorean triples. There is conjecture that one of the columns describes a systematic algorithm or strategy for their construction.

4. Thales of Miletus
   (622-547 bc, Miletus (modern Turkey)) Thales is given credit for many geometric propositions such that that the angle inscribed in a semicircle is right. He was also an applied mathematician and used geometric principles to solve problems like determining the distance of a ship a sea.

5. Pythagoras of Samos
   (585-500 bc, Samos, an island in the Aegean) Pythagoras founded a school that made many contributions to mathematics including for example the study of figurative numbers, a subject of continued interest. Most important, he is given credit for the discovery that the ratio of the side of a square and the diagonal of a square can never be expressed as a rational number. The discovery of incommensurable quantities meant that the study of numbers and the study of geometry became separated.

6. Euclid
   (323-285 bc, Alexandria Egypt) Euclid wrote a mathematical text called Elements. In book X of Elements, Euclid describes a formula for constructing every possible Pythagorean triple. While it is clear that his method produces Pythagoran triples, it is not clear Euclid understood that every Pythagoran triple would take this particular form.

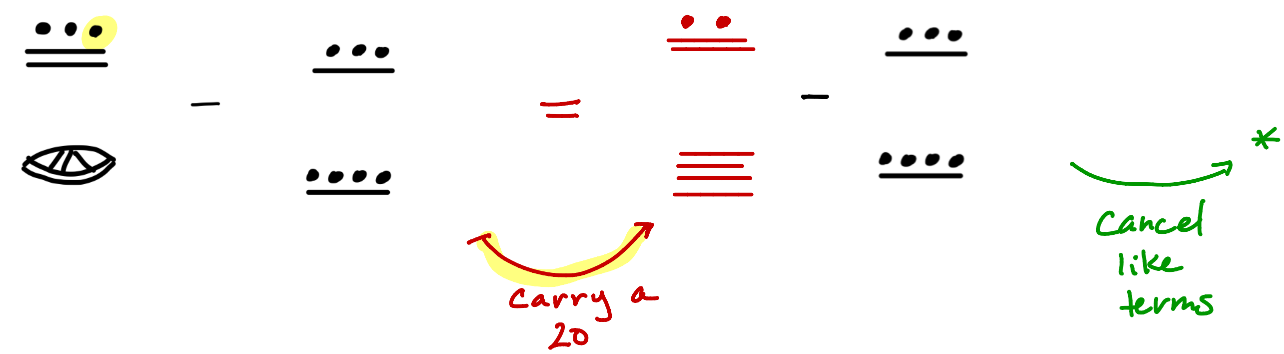
7. Theon of Smyrna (130 ce, Smyra (modern Turkey)) Theon constructed two sequences of numbers (so called ”side” and ”diagonal” numbers) such that their quotients produces better and better approximations of $\sqrt{2}$. The sequences were recursively defined and can be shown to converge to $\sqrt{2}$. Theon's strategy can be used to obtain converging sequences for the roots of other numbers, too.

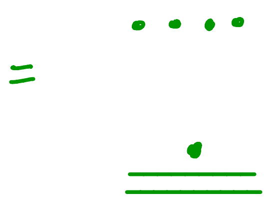
8. Eudoxus of Cnidos
   (408-355 bc, Cnidos (in modern Turkey)) Eudoxus developed a theory that allowed the comparisons of ratios of like quantities. The theory allowed ancient Greek mathematicians to avoid the defining or even working with irrational numbers. It was a way of managing the exists of incommensurable quantities. Euclid incorporated Eudoxus' work in the Elements. The ability to compare otherwise incommensurable magnitudes allowed the pursuit theoretical, rigorous mathematics in geometry.

Part II

For this part, you may use a calculator and up to two pages of notes. This part is worth 75 points out of 100 total points.

1. (8 points) Demonstrate *using Mayan symbols* how ancient Mayans must have performed subtraction below.





Check:

$(13\cdot 20) - (8\cdot 20 + 9)$

$= 91 = 4\cdot 20 + 11$ ✓

2. (8 points) Use the ancient Egyptian method to divide 47 by 18.

thinking

want x so that

$x \cdot 18 = 47$

$47 - 36 = 11$

$11 - 9 = 2$

answer:

	1	18
→	2	36 ✓
→	$\frac{1}{2}$	9 ✓
→	$\frac{1}{9}$	2 ✓
answer →	$2+\frac{1}{2}+\frac{1}{9}$	47

Check:

$\frac{47}{18} = 2.6\overline{1}$

$= 2+\frac{1}{2}+\frac{1}{9}$ ✓

3. (10 points)

   (a) Write the base 60 number $(3, 45)$ in base 10.

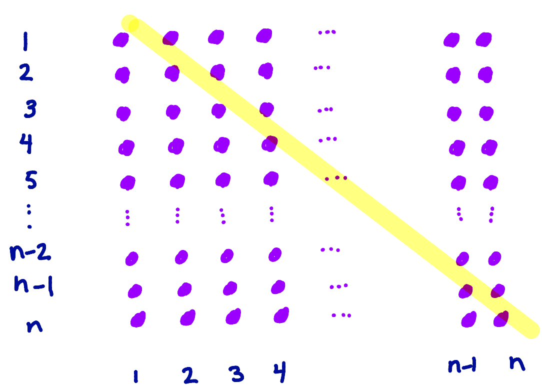
   $(3, 45)_{60} = 3 \cdot 60 + 45 = 225.$

   (b) Explain how $(3, 45)$ could be viewed by ancient Babylonians as the reciprocal of 16. Your answer should include a computation.

   Since $16 * 225 = 3600 = 60^2 = (1, 0, 0)_{60}$, if one chooses to shift the powers of 60 so that instead of $(3, 45)_{60} = 3 \cdot 60 + 45$, we view $(3, 45)_{60} = 3 \cdot 60^{-1} + 45 \cdot 60^{-2}$ then the product of 16 and $(3, 45)$ is indeed 1. Without a sexagesimal point, ancient Babylonians had to interpret the magnitude of their numerical representation by context.

4. (9 points) Give a proof-by-picture proof that the sum of two consecutive triangular numbers is a square number.

Show $t_{n-1} + t_n = S_n$



The purple dots represent S_n as it is $n \times n$.

Above the yellow line is t_{n-1}.

Below the yellow diagonal is t_n.

5. (10 points) Solve the problem below using the method of false position.

   A quantity, its half, and its fifth added together becomes two. What is the quantity?

thinking:

Solve

$x + \frac{x}{2} + \frac{x}{5}$

Check:

$\frac{20}{17} + \frac{10}{17} + \frac{4}{17}$

$= \frac{34}{17} = 2$ ✓

Answer:

Guess the quantity is 10.

Then $x + \frac{x}{2} + \frac{x}{5} = 10 + 5 + 2 = 17$.

Since we make 17 into 2 by scaling by $\frac{2}{17}$,

we scale our guess by the same factor.

Answer: $\frac{10 \cdot 2}{17} = \frac{20}{17}$

6. (30 points) Short Answer

(a) Give an example of a numerical system of representation that is *additive* and another example that is *positional* and explain the difference.

The ancient Egyptian heiroglyphic numerical system was additive. Various symbols represented powers of 10 which were added up to determine the number. The order in which symbols were written didn't affect the number being represented. The Mayans developed a fully positional system including a symbol for zero. Their system only used three total symbols (dots, lines, shell) and the position of the symbols determined what power of 20 (or 20 and 18) was being represented.

(b) Greek mathematicians starting using an *alphabetic* or *ciphered* numeral system by the 5th century b.c. Explain what an alphabetic numeral system is, its advantages and disadvantages.

An alphabetic or ciphered system takes letters used in regular language and associates with each letter a number. One of the advantages is that it is compact, unlike for example Egyptian hieroglypics, and a familiar kind of writing. One of the disadvantages is that it requires a lot of memorization or large tables for arithmetic.

(c) Describe two impressive accomplishments of ancient Egyptian mathematicians.

In the Moscow papyrus, Egyptians demonstrated many correct geometric formulas and one of the most impressive was a correct formula for the volume of a truncated pyramid. While is was a choice of Egyptians to restrict the representation of fractions to unit fractions, they nevertheless demonstrated great facility with this system. Their doubling algorithm for multiplication was wonderfully quick and easy, at least for integers.

(d) Compare the Babylonian approach to quadratic equations with the modern approach.

Without the sophisticated algebraic representation we use today, Babylonians had to describe their problems in a sort of essay style. Often the essay-style description is not obviously even a quadratic equation but requires a lot of manipulation to reform it as such. Without negative numbers or a formal zero, Babylonians could not arrange every quadratic problem into a common form. Thus, Babylonians had multiple types of quadratic equations each with its own strategy for solution.

(e) Explain what **ancient Greeks mathematicians** meant by the statement "the diagonal of a square and its side are incommensurable."

They meant that it is not possible to find any measure (or smaller line segment) that will divide both the side and the diagonal using integer multiples. Another way to say it is that no matter what measure one chooses, either the side or the diagonal (or both!) will require a fraction of the chosen measure.