Solutions

Part I

1. Rhind and Moscow papyri (time approximately: 2000-1000 BC)
   These documents provide examples of early Egyptian mathematics and effectively includes the correct formula for the volume of a truncated pyramid given the lengths of base, length of top, and vertical height.

2. Pythagoras (time approximately 550 BC)
   He was a Greek mathematician who founded a mathematical community in what is now southern Italy. Among other things, this group discovered and proved that $\sqrt{2}$ is irrational.

3. Claudius Ptolemy (time approximately: 100's AD)
   Ptolemy was a Greek astronomer and mathematician located in Alexandria who wrote a very influential text called *Almagest* that included a table of chords for angles from $1/2^o$ to 90^o in increments of $1/2^o$. This document eclipsed all earlier Greek works on astronomy.

4. Euclid (time approximately: 300's BC)
   Euclid was a Greek mathematician located in Alexandria who wrote a very influential text called *The Elements* which included results on 2- and 3-dimensional geometry, number theory, and proportions. Its axiomatic approach and organization have been held as a sort of ideal of mathematical thought for centuries.

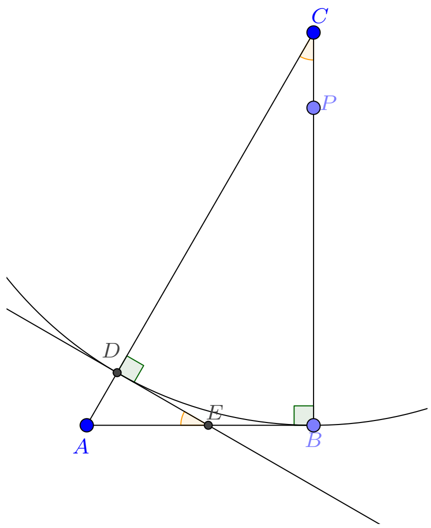
5. Archimedes (time approximately: 200's BC)
   He was a Greek engineer, scientist and mathematician located in Syracuse. He wrote *On the Sphere and the Cylinder* in which he proves that the volume and surface area of the sphere of radius r is $2/3$ those of a cylinder of radius r and height $2r$. In addition to proving the stated theorems, he attempted to give intuition to the result by a sort of theoretical experiment involving slicing and weighing the objects.

6. Mahavira (time approximately: 850 AD)
   An Indian mathematician who was one of the first to treat the symbol for zero as a full-blown number on par with 1, 2, 3 and so forth and not just a placeholder. In particular, he attempted to define addition, subtraction, multiplication and division with zero.

7. Eudoxus (time approximately: 400 BC)
   Eudoxus was a Greek mathematician from Athens who resolved the $\sqrt{2}$ crisis by constructing a new definition of comparable ratios. His definition essentially used a three-fold comparison: $<$, $=$, and $>$ to rational numbers.

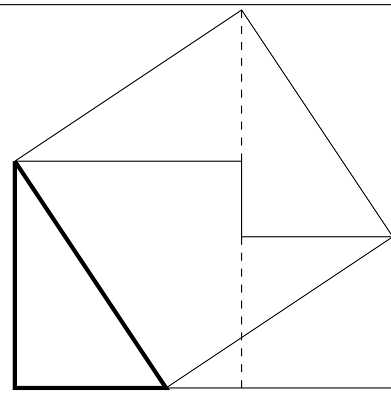
Part II

1. (8 points) Give a geometric argument that $\sqrt{3}$ is irrational.



- Proceed by contradiction and assume $\sqrt{3}$ is rational (or commensurable).
- Construct $\triangle ABC$ such that $2AB = AC$ and $\angle B = 90^o$.
- From this construction, $BC = \sqrt{3}$.
- Since we are assuming $\sqrt{3}$ is commensurable or rational, we can find a length, CP, that is a common measure of AB and BC.
- Since $2AB = AC$, CP is a measure of AC, too.
- Construct D on AC such that $CD = CB$.
- Since CP measures CB, it measures CD.
- Since CP measures CA and CD, it measures DA.
- Construct E on AB such that ED is orthogonal to AC (or alternatively such that ED is tangent to the circle with center C and radius BC.)
- Now $\triangle ABC$ is similar to $\triangle ADE$ by angle-angle-angle which implies that $AE = 2DA$.
- Since CP measures DA, it must measure EA.
- SInce CP measures AE and AB it must measure EB.
- By construction $ED = BE$ which implies CP measures DE.
- Thus, CP measures all three sides of the smaller 30-60-90 triangle.
- By repeating this process, we have shown that CP is a measure of arbitrarily small line segments, a contradiction.

2. (8 points) The figure below suggests a congruency-by-addition proof of the Pythagorean Theorem. Fill in the details of the proof. *You must carefully state the geometric facts you are using.* I would suggesting beginning by labelling the figure.



See scan at end.

3. (8 points each)

   (a) Find, in the Egyptian fashion, the quotient $184\,\overline{2} \div 15$. [Note that while you may choose to reverse-engineer your answer, your official work and answer should look Egyptian.]

   (b) An ancient Babylonian tablet solves the division problem

   $$1,\ 10 \div 45$$

   in the following manner:

   $$(1,\ 10) \times (1,\ 20) = 1,\ 33,\ 20.$$

   Show that this answer is in fact correct and explain the reasoning behind the approach.

See scan at end.

4. (8 points each) Answer each question below. All answers should consist of at least one full coherent sentence. Be specific.

   (a) What is the difference between experimental geometry and deductive geometry?
   Both result in formulas for calculating areas, perimeters, volumes, surface areas of geometric objects. What is different is *how* these formulas are derived. In experimental geometry it is sufficient to build a model and observe that the formula appears to work in this model. Deductive geometry requires a formal proof that the formula is correct and requires a mutually agreed upon logical framework in which to make this argument. In deductive geometry, examples – no matter how plentiful or persuasive – are not sufficient to establish the correctness of a formula.

   (b) What is the difference between a proof using commensurables and one using the Eudoxian method?
   Both are proof techniques used by ancient Greek mathematicians to establish relative proportions of geometric figures. Proofs using commensurables assumes that given any two magnitudes (length, angle measure, etc) there must exist some third magnitude (length, angle measure, etc) that measures the first two an even number of times. Another way to say it is that proof by commensurables will compare two figures by first

asserting that they can be split into *whole* pieces of *equal* magnitude. The comparison becomes merely comparing the number of these equal pieces. The problem, we know, is that their hypothesis was false. One is not guaranteed that third common measure! The Eudoxian method compares two magnitudes by considering integer *multiples* of them. By this definition, one can assert two pairs magnitudes are in equal ratio by showing multiples forcing a greater than (or equal or less than) results in the same greater than (or equal or less than) relationship.

(c) Discuss the significance and impact of Euclid's *Elements.*
The remarkable thing about Euclid's Elements was not the content but the structure. Each book began with axioms, postulates and definitions and from these, Euclid built one theorem (proposition) after another in a formal, logically sound manner. There was no motivation or explanation beyond the proofs. This rigorous, minimal style became emblematic of how mathematics should be written.
It is difficult to overstate its significance. The fact that there is little known about geometry prior to Euclid is evidence of its superiority to all the came before it. Euclid's choice of axioms drove mathematical research for hundreds of years after its first appearance. The role of proof in the teaching and learning of geometry remains a subject of debate today.

(d) Discuss the evolution of the Hindu-Arabic base 10 positional system.
Numerical representation in base 10 began pre-history. While it is not certain precisely when a fully-formed base 10 *positional* system began, it is clear that by 700 AD Indian mathematicians were regularly using such a system. By 775 AD this system had made its way into the repertoire of Islamic mathematicians. By 1100's, Islamic mathematics was being translated into Hebrew and Latin in southern Europe. By 1202, Fibonacci is actively promoting the superiority of a base 10 positional system over that of Roman numerals. While the role of zero (whether number or place holder) varies widely, we know that at least by 850 AD there existed Indian mathematicians who were treating zero as a full-fledged number. A best guess as to what inspired a base 10 positional system would be the abacus or counting boards.

(e) What originally motivated what we now call the sine function and how was the original concept different from our present one?
Our modern version of the sine function associates angles with ratios of sides of right triangles or with points on the unit circle in the xy-plane. The original form involved associating central angles of circles of some fixed radius with chord lengths of the circles. Moreover, the radius of the circle being measured varied quite a bit depending on the mathematician constructing the table. Another difference is that advanced mathematicians tend to measure angles in radians as opposed to the original choice of measurement, degrees. These measurements were motivated by astronomers attempting to understand and predict the movements of heavenly bodies.