MATH 341 LINEAR ALGEBRA
FALL 2025
MIDTERM 1

Name: Solutions **Time: 60 minutes**

Instructions: Show all work for full credit. Partial credit will be awarded for correct methods even if the final answer is incorrect. No book, notes, electronics, calculator, or internet access is permitted.

problem	points	score
1	20	
2	10	
3	20	
4	20	
5	30	
total	100	

(1) (20 points) Consider the following system $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 5y + 2z &= 4 \\ x + 4y + 4z &= -1 \end{aligned}$$

(a) Solve the linear system by *elimination* and then *back-substitution* on the *augmented matrix*. Use the standard algorithm.

$$\begin{array}{c} \begin{matrix} 2 & 4 & -2 & 6 \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 2 & 4 \\ 1 & 4 & 4 & -1 \end{array}\right] \end{array} \xrightarrow[\ell_{31}=1]{\ell_{21}=2} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -2 \\ 0 & 2 & 5 & -4 \end{array}\right] \\ \begin{matrix} 2 & 8 & -4 \end{matrix} \end{array} \xrightarrow{\ell_{32}=2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$-3z = 0$. So $z = 0$

$y = -2 - 4z = -2$. So $y = -2$

$x = 3 + 4z - 2y = 3 - 2(-2) = 7$ So $x = 7$

answer: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$

(b) List the pivots.

$1, 1, -3$

(c) Using your work in part(a), determine the LU-factorization of the matrix A.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & -3 \end{bmatrix}$$

(d) Suppose the vector $\mathbf{b} = (3, 4, -1)$ was replaced with the vector $\mathbf{c} = (3, 3, 3)$, would the system $A\mathbf{x} = \mathbf{c}$ have a solution? (You should answer this question without doing any additional work.)

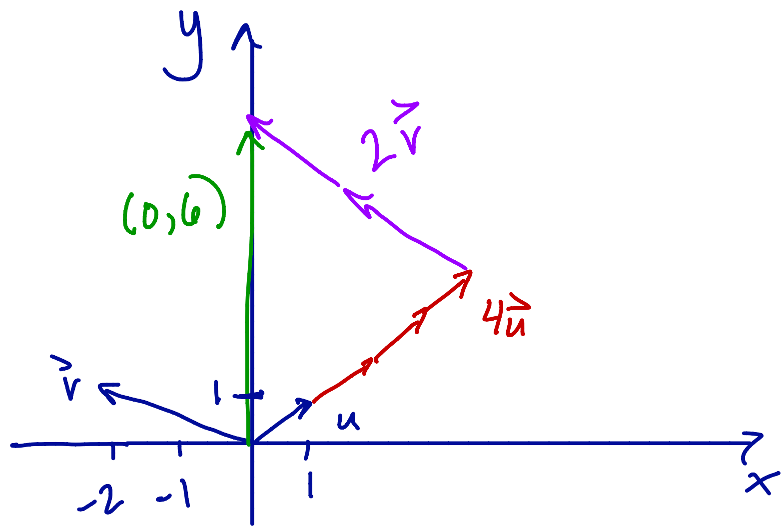
Yes. $A\vec{x} = \vec{c}$ has a solution. The LU-factorization can find it for us.

(2) (10 points) The system of equations $A\mathbf{x} = \mathbf{b}$ below has solution $(x, y) = (4, 2)$.

$$x - 2y = 0$$
$$x + y = 6$$

(a) Draw the *column view* of this solution.

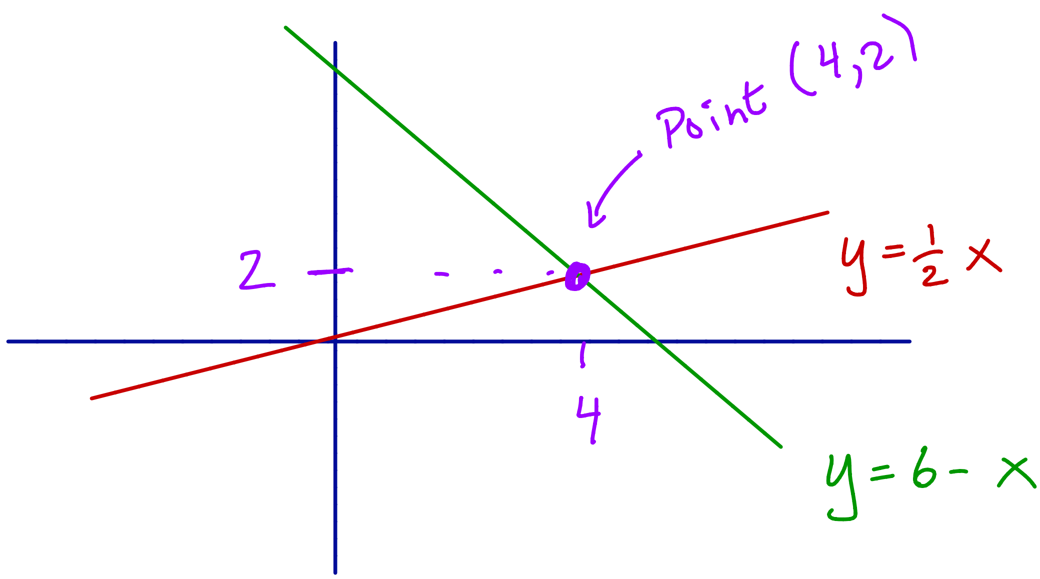
$$4 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{u}} + 2 \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



(b) Draw the *row view* of this solution.

(4,2) is the point of intersection of

$y = \frac{1}{2}x$ and $y = 6 - x$



(3) (20 points)

(a) *Use Gauss-Jordan elimination* to find the inverse of the matrix $A = \begin{bmatrix} 2 & -3 \\ -4 & 8 \end{bmatrix}$.

$$\begin{array}{cccc} -4 & +6 & -2 & 0 \end{array}$$

$$\left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ -4 & 8 & 0 & 1 \end{array}\right] \xrightarrow{\ell_{21} = -2} \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{array}\right] \xrightarrow{\ell_{12} = -\frac{3}{2}} \left[\begin{array}{cc|cc} 2 & 0 & 4 & 3/2 \\ 0 & 2 & 2 & 1 \end{array}\right]$$

$$0 \quad -3 \quad -3 \quad -3/2$$

$$\xrightarrow[\text{2}]{\text{divide both rows by}} \left[\begin{array}{cc|cc} 1 & 0 & 2 & 3/4 \\ 0 & 1 & 1 & 1/2 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 2 & 3/4 \\ 1 & 1/2 \end{bmatrix}$$

(b) Show that your answer is correct.

$$\begin{bmatrix} 2 & -3 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 2 & 3/4 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} 4-3 & \frac{3}{2} - \frac{3}{2} \\ -8+8 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) Use your answer in part (a) to solve the system:

$$2x - 3y = 1$$
$$-4x + 8y = -1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3/4 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 - \frac{3}{4} \\ 1 - 1/2 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 1/2 \end{bmatrix}$$

(4) (20 points) Consider the vectors $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

(a) (5 points) Is the angle between $\mathbf{v}$ and $\mathbf{w}$ acute, obtuse, or right? Justify your conclusion.

$$\vec{u} \cdot \vec{v} = 0 \cdot 1 + (-1)(4) + (3)(-1)$$
$$= -4 - 3 = -7 < 0$$

angle is obtuse

(b) (5 points) Find a unit vector, $\mathbf{u}$, in the same direction as vector $\mathbf{v}$.

$$\|u\| = \sqrt{0^2 + (-1)^2 + 3^2} = \sqrt{10}$$

$$\vec{a} = \left(0, \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

(c) (5 points) Find a nonzero vector, $\mathbf{x}$, orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

Want $\vec{x} = (x, y, z)$ so that

$\vec{x} \cdot \vec{u} = -y + 3z = 0$. So $\boxed{y = 3z}$ and

$\vec{x} \cdot \vec{v} = x + 4y - z = 0$. So $x = z - 4y = z - 4(3z) = -11z$.

(randomly) pick $z = 1$. So $x = -11$, $y = 3$.

ANSWER: $\vec{x} = (-11, 3, 1)$.

(d) (5 points) Is $\mathbf{c} = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$ a linear combination of $\mathbf{v}$ and $\mathbf{w}$? Justify your conclusion.

Yes. By inspection we can see:

$$1\vec{v} + 2\vec{w} = \vec{c}.$$

(5) Short Answer (5 points each for 30 points)

(a) In the system of equations

$$\begin{aligned} x - 3y &= 5 \\ 4x + 12y &= b, \end{aligned} \qquad \cdot 4$$

determine a value for b for which the system has

(i) no solution. $b = 0$ ← or anything that isn't 20

(ii) an infinite number of solutions. $b = 20$

(b) Give the 4 by 4 elimination matrix, E_{42}, that would subtract 5 times row 2 from row 4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -5 & 0 & 1 \end{bmatrix}$$

(c) Give an example of a nonzero 2 by 2 matrix A such that A^2 is the zero matrix.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(d) Determine the second column of the matrix product, AB, below:

$$AB = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 4 & 4 \\ 1 & 0 & -1 & -1 & -1 \\ 0 & 3 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 2 & -1 & 4 \\ 1 & 0 & 1 \\ 3 & 0 & 4 \\ -2 & 0 & 0 \end{bmatrix}.$$

$(4\times 5)(5\times 3) = 4\times 3$

$$\mathrm{col}_2(AB) = A\cdot \mathrm{col}_2(B) = A\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 1-2 \\ 1-0 \\ 0-3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ -3 \end{bmatrix}$$

(e) Is the matrix below invertible? Justify your conclusion.

$$\begin{bmatrix} -2 & 1 & 8 \\ 4 & -2 & -16 \\ 1 & 1 & 1 \end{bmatrix}$$

No. row 2 is a multiple of row1.

(f) Suppose that matrices A and B are symmetric and have the same dimensions. Show that ABA must also be symmetric.

A and B symmetric means $A^T = A$ and $B^T = B$.

Now $(ABA)^T = A^T (AB)^T$

$= A^T (B^T A^T)$

$= ABA$.