MATH 314 LINEAR ALGEBRA
FALL 2025
MIDTERM 2

Name: Solutions **Time: 60 minutes**

Instructions: Show all work for full credit. Partial credit will be awarded for correct methods even if the final answer is incorrect. No book, notes, electronics, calculator, or internet access is permitted.

problem	points	score
1	16	
2	15	
3	15	
4	15	
5	16	
	+ 3 EC	
6	23	
total	100	

(1) (16 pts) Below is the matrix **A** and its reduced row echelon form, **U**.

$$A = \begin{bmatrix} 2 & -6 & 0 & 4 & 0 \\ -1 & -3 & 2 & -4 & 0 \\ 2 & -6 & 2 & 2 & 0 \\ 1 & -3 & 4 & -2 & 2 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (4 pts) Find a **basis** for the row space of **A**, $C(\mathbf{A}^T)$, and state its **dimension**.

$$\begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{dim} = 3$$

(b) (4 pts) Find a **basis** for the column space of **A**, $C(\mathbf{A})$, and state its **dimension**.

$$\begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \qquad \text{dim} = 3$$

(c) (6 pts) Find a **basis** for the null space of **A**, $N(\mathbf{A})$, and state its **dimension**.

$$x_2 = 1: \; x_4 = 0 \; \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_4 = 1: \; x_2 = 0 \; \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{dim} = 2$$

(d) (1 pt) State the **dimension** of the left null space of **A**, $N(\mathbf{A}^T)$.

$$\dim(N(A^T)) = 4 - 3 = 1$$

(e) (2 pts) Does the left null space contain *any* nonzero vector? If so, find one. If not, explain why.

Yes. (b/c dim = 1)

Need a vector orthogonal to $C(A)$. $\vec{z} = \begin{bmatrix} 3/2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ (by inspection w/)

(2) (15 pts) Consider the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 2x_1 + 3x_2 + 3x_3 \phantom{{}- x_4} &= 9 \\ 2x_1 + 6x_2 + x_3 - x_4 &= 27 \\ -4x_1 - 3x_2 - 7x_3 - x_4 &= -3 \end{aligned}$$

Here is the row-reduced echelon form of the augmented matrix:

$$[A\ \ \mathbf{b}] \qquad \longrightarrow \qquad [R\ \ \mathbf{d}] = \begin{bmatrix} 1 & 0 & 0 & 1/2 & 3 \\ 0 & 1 & 0 & 1/3 & 4 \\ 0 & 0 & 1 & 0 & -3 \end{bmatrix}$$

What is the general solution of the system? Show your work.

Find x_p: Set $x_4 = 0$. $\quad x_p = \begin{bmatrix} 3 \\ 4 \\ -3 \\ 0 \end{bmatrix}$

Find x_n: $\begin{bmatrix} -1/2 \\ -1/3 \\ 0 \\ 1 \end{bmatrix}$

General Soln: $x = \begin{bmatrix} 3 \\ 4 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ -1/3 \\ 0 \\ 1 \end{bmatrix}$

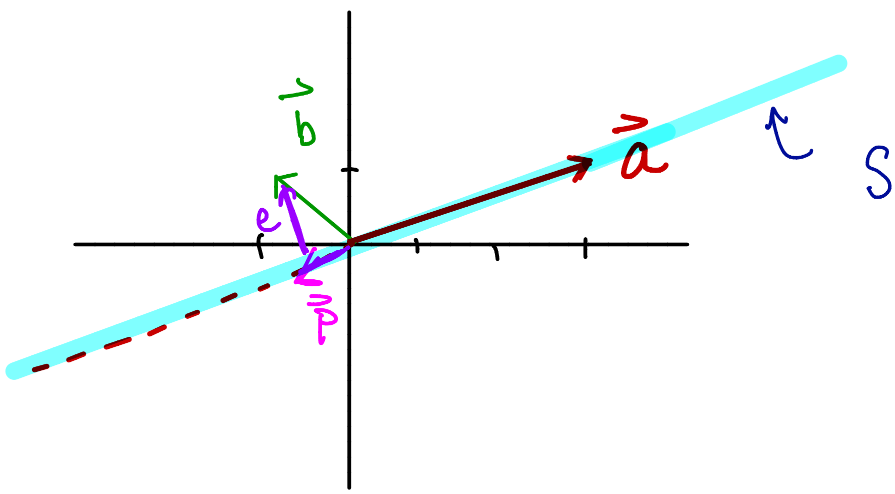
(3) (15 pts) Let S be a 1-dimensional subspace of $\mathbb{R}^2$ spanned by the vector $\mathbf{a} = (3, 1)$.

(a) (6 pts) Find the projection, $\mathbf{p}$, of $\mathbf{b} = (-1, 1)$ onto S and the error, $\mathbf{e}$.

$$p = \left(\frac{b \cdot a}{a \cdot a}\right) a = \frac{-3+1}{9+1}(3,1) = \frac{-2}{10}(3,1) = \left(-\frac{3}{5}, \frac{-1}{5}\right)$$

$$e = b - p = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -3/5 \\ -1/5 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 6/5 \end{bmatrix}$$

(b) (4 pts) On the same set of axes, draw and label $\mathbf{s}$, $\mathbf{b}$, $\mathbf{p}$, and $\mathbf{e}$.



(c) (5 pts) Compute P, the matrix of the projection.

$$P = a(a^T a)^{-1} a^T = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3, 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 10 \end{bmatrix} \begin{bmatrix} 3, 1 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

(4) (15 pts) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by:

$$\mathbf{a}_1 = (1,1,0,0),\ \mathbf{a}_2 = (0,2,1,0),\ \mathbf{a}_3 = (1,1,3,1)$$

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = r_1$$

$$\boxed{q_1 = \frac{1}{\sqrt{2}}(1,1,0,0)}$$

$$r_2 = a_2 - \left(\frac{a_2 \cdot r_1}{r_1 \cdot r_1}\right) r_1$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{q_2 = \frac{1}{\sqrt{3}}(-1,1,1,0)}$$

$$r_3 = a_3 - \left(\frac{a_3 \cdot r_1}{r_1 \cdot r_1}\right) r_1 - \frac{a_3 \cdot r_2}{r_2 \cdot r_2} r_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{(-1+1+3)}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{q_3 = \frac{1}{\sqrt{7}}(1,-1,2,1)}$$

orthonormal basis: q_1, q_2, q_3

(5) (16 pts) An experiment produced the following data points in the (x, y) plane:

$$(1, 6),\ (3, 3),\ (4, 1)$$

We would like to find the line, $y = mx + b$, of best fit.

(a) (3 pts) Write down the system of equations we would *like* to solve in order to find m and b. That is, this system would have a solution if all points were on a common line.

$$6 = m \cdot 1 + b$$
$$3 = m \cdot 3 + b$$
$$1 = m \cdot 4 + b$$

(b) (2 pts) Rewrite the system of equations in part (a) in vector form. (We typically write this as $\mathbf{Ax} = \mathbf{b}$) So, state the coefficient matrix $\mathbf{A}$, the variable vector $\mathbf{x}$ and the constant vector $\mathbf{b}$.

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$$
$$A \cdot \vec{x} = \vec{b}$$

(c) (2 pts) Is your vector $\mathbf{b}$ in the column space of your matrix $\mathbf{A}$? Give a very brief explanation.

No. The points are not in a line. So there is no solution.

(d) (6 pts) Write down the system of equations we will actually solve in order to find the line of best fit. These are also called the *normal* equations.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 26 \end{bmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 6+9+4 \end{pmatrix} = \begin{pmatrix} 10 \\ 19 \end{pmatrix}$$

$$\begin{array}{r} 16 \\ +\ 9 \\ +\ 1 \\ \hline 26 \end{array}$$

$6+9+4$

$$\begin{pmatrix} 3 & 8 \\ 8 & 26 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 10 \\ 19 \end{pmatrix} \quad \text{or}$$

$$3b + 8m = 10$$
$$8b + 26m = 19$$

problem 5 continued on the next page ... $\longrightarrow$

(e) (3 pts) A solution to the normal equations produces the slope and y-intercept of the line of best fit. In what sense is it *best*? That is, what quantity is minimized? You should be able to give a mathematical formula here.

$\|\vec{e}(x)\|^2 = \|A\vec{x} - \vec{b}\|^2$ is smallest when $\vec{x}$ is $\hat{x}$.

You could also write:

$(b+m-6)^2 + (b+3m-3)^2 + (b+4m-1)^2$ is smallest when $(b,m) = \hat{x}$.

(f) (3 pts extra credit) We typically write the least squares solution as $\widehat{\mathbf{x}}$, which in this case is $(7.7, -1.6)$. Translate this least squares solution into a "projection" problem. That is, somehow $\widehat{\mathbf{x}}$ should tell you how to find the projection, $\mathbf{p}$, of some vector $\mathbf{b}$ onto some subspace S. What are $\mathbf{p}$, $\mathbf{b}$ and S?

$\vec{b} = (6,3,1)$

$$\vec{p} = A\hat{x} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 7.7 \\ -1.6 \end{pmatrix} = \begin{bmatrix} 7.7 - 1.6 \\ 7.7 - 3(1.6) \\ 7.7 - 4(1.6) \end{bmatrix}$$

$S = C(A)$

(6) (40 pts) Short Answer

(a) (3 pts) Do the vectors in $\mathbb{R}^3$ whose entries add up to 1 form a subspace of $\mathbb{R}^3$? Justify your conclusion.

No. $\vec{0} = (0,0,0)$ is not in the set.

(b) (3 pts) What does it mean for the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ to be *linearly independent*? (Give a precise mathematical explanation.)

The only solution to

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 = \vec{0}$$ is

for $c_1 = c_2 = c_3 = c_4 = 0$.

alt. Soln. $P = A(A^TA)^{-1}A^T$. So $P^2 = (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T) = A(A^TA)^{-1}A^T = P$ ✓

$\underbrace{A^TA}_{I_n}$

(c) (4 pts) Suppose that $\mathbf{P}$ is a projection matrix. What is $\mathbf{P}^2$ and why?

$P^2 = P$ This is because projecting a vector in a subspace to the subspace does not change the vector.

(d) (3 pts) Briefly explain why the row space and the null space of the matrix $\mathbf{A}$ are orthogonal.

If x in $N(A)$, then $Ax = \vec{0}$. This means $\text{row}_i(A) \cdot \vec{x} = 0$ for all rows. So $\vec{x} \perp$ all rows in $C(A^T)$.

(e) (3 pts) Determine if the following statement is true or false. Justify your conclusion.

If B is not invertible, then AB is not invertible.

True. $\det(B) = 0 \iff B$ not invertible

Now $\det(AB) = \det(A)\det(B) = \det(A) \cdot 0$.

So AB is not invertible.

(f) (3 pts) Suppose E_{ij} is an elimination matrix. Is it possible to determine $\det(E_{ij})$? Explain your answer briefly.

$\det(E_{ij}) = 1$ because E_{ij} is lower triangular with 1's on the diagonal

(g) (4 pts) Suppose $\mathbf{A}$ is a 3×4 matrix and the vector $\mathbf{a}$ forms a *basis* for the null space of $\mathbf{A}$.

(i) What is the rank of $\mathbf{A}$? 3

(ii) What, if anything, can you determine about solutions to the system of equations $\mathbf{Ax} = \mathbf{b}$? Explain your answer.

Since A has full row rank and a free column, $Ax = b$ will always have an infinite number of solutions