1. Consider the vectors

$$\mathbf{a} = (1, -1, 3), \quad \mathbf{b} = (2, 1, 1)$$

   (a) Are $\mathbf{a}$ and $\mathbf{b}$ orthogonal?

   (b) Give a third vector $\mathbf{c}$ that is orthogonal to $\mathbf{a}$ and one unit long.

   (c) How many correct answers are there to part (b)?

2. Consider the equation

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

   (a) Sketch (approximately) the column picture for this equation.

   (b) In a sentence or two, explain how this picture indicates this equation is solvable.

   (c) Change only one entry in the matrix so that the equation cannot be solved.

3. Solve the following, using elimination on the augmented matrix, and back substitution:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -2 \\ 0 & -6 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}.$$

4. (a) Give the two 3×3 elimination matrices that put the above system in upper-triangular form.

(b) Give *one* matrix that does both these elimination steps at once. (Make sure you think about the correct order of the steps.)

5. What should z be to make the the (2,3)-entry of following matrix product be 0?

$$\begin{pmatrix} 1 & 2 & z \\ 2 & z & 3 \\ z & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

6. Find A^{-1}, or show it doesn't exist, for

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ -1 & 2 & -1 \end{pmatrix}.$$

7. Give an LU factorization of

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -4 & -3 & 5 \\ 0 & -1 & 4 \end{pmatrix}.$$

8. Suppose the LU factorization of a matrix A is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Use this to solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (-1, 6, -3)$ by solving two triangular systems. Do NOT compute A to do this problem!

9. Here are three equations in two unknowns:

$$\begin{aligned} 2x + 2y &= 6 \\ x - 3y &= -1 \\ 4x + y &= 0 \end{aligned}$$

We can also write this as $A\mathbf{x} = \mathbf{b}$ where A is a 3×2 matrix and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$ are vectors.

(a) Sketch the "row picture": sketch each equation as a line in the (x, y) plane. Do they intersect?

(b) Sketch the "column picture": sketch each column of A, and also $\mathbf{b}$, in three-dimensional space. Will you be able to find a linear combination of the columns of A which gives $\mathbf{b}$?

(c) Change one entry of the right side so that the linear system does have a solution, and find that solution.

10. (a) Consider the new linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix},$$

and $\mathbf{x} = (x_1, x_2, x_3)$ is unknown (*for now*). Sketch the row picture, that is, sketch each of the three equations as a plane. Note it is easier to sketch each plane on separate axes; show three sketches.

(b) What is the solution of the system in part **(i)**?

11. Consider the following linear system $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 2x_1 + x_2 - 9x_3 &= -6 \\ 4x_1 - 3x_2 - 21x_3 &= -20 \\ -6x_1 - 13x_2 + 24x_3 &= 5 \end{aligned}$$

Solve the linear system by *elimination* and then *back-substitution*. Use the standard algorithm. Show your work, and in particular show, as an intermediate stage, the triangular system which you get after elimination.

12. From problem above,

    (a) what elimination matrices $E_{21}, E_{31}E_{32}$ did the row operations?

    (b) What are the inverses $L_{21} = E_{21}^{-1}$, $L_{31} = E_{31}^{-1}$, $L_{32} = E_{32}^{-1}$?

    (c) What numbers were the pivots? What is the determinant of A?

    (d) The computation can regarded as factoring $A = LU$. What lower triangular matrix L and upper triangular matrix U were computed?

    (e) Multiply LU and confirm you get the original matrix A.