(1) The ABC Bike Company has two locations, Downtown and Beach, where people can rent a bike for a day. The bikes can be returned at either location. The table below shows where people tend to return the bikes given where they are rented.

rented downtown	80% returned downtown	20% returned at the beach
rented at the beach	30% returned downtown	70% returned at the beach

(a) Suppose they start the beginning of the week (say Monday) with half of their bikes downtown and at the beach. Calculate the distribution of the bikes between the locations at the end of Monday. Give a detailed calculation.

D: $(0.8)(\frac{1}{2}) + 0.3(\frac{1}{2}) = 0.4 + 0.15 = 0.55$

B: $(0.2)(\frac{1}{2}) + 0.7(\frac{1}{2}) = 0.1 + 0.35 = 0.45$

(b) Use your answer above to determine the distribution of bikes at the end of the day on Tuesday assuming they do not redistribute the bikes Monday night.

D: $(0.8)(0.55) + (0.3)(0.45) = 0.575$

B: $(0.2)(0.55) + (0.7)(0.45) = 0.425$

(c) Find a matrix $\mathbf{A}$ such that if $\mathbf{x} = (x_d, x_b)$ is a vector with the distribution of bikes between downtown (x_d) and the beach (x_b) at the beginning of the day, $\mathbf{Ax}$ will give the distribution at the end of the day. Show your answer is correct.

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_d \\ x_b \end{bmatrix} = \begin{bmatrix} 0.8x_d + 0.3x_b \\ 0.2x_d + 0.7x_b \end{bmatrix} \quad \checkmark$$

(d) Use a computational tool to determine the distribution of bikes at the end of month assuming the company does not redistribute any bikes. Show the computation below. What do you think happens to the distribution of bikes long term?

$$A^{30} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.6000 \\ 0.4000 \end{bmatrix}$$

The distribution settles around 60 : 40 :: D : B.

(e) Recall that we determined earlier that the matrix A has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1/2$, with associated eigenvectors $\mathbf{x}_1 = (.6, .4)$ and $\mathbf{x}_2 = (1, -1)$. Show that the starting distrubution $\mathbf{x}_0 = (0.5, 0.5)$ can be written as $\mathbf{x}_1 - (0.1)\mathbf{x}_2$.

$$1\begin{bmatrix} .6 \\ .4 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ .5 \end{bmatrix} \quad \checkmark$$

(f) Use part (e) to confirm your conclusion in part (d).

$\lambda_2 = \frac{1}{2}$

* Note $(\frac{1}{2})^{30} \approx 9.3 \times 10^{-10}$

$$A^{30}\begin{bmatrix} .5 \\ .5 \end{bmatrix} = A^{30}\left(\begin{bmatrix} .6 \\ .4 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = A^{30}\begin{bmatrix} .6 \\ .4 \end{bmatrix} - \frac{1}{10}A^{30}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} - \frac{1}{10}\cdot\left(\frac{1}{2}\right)^{30}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_1 = 1$

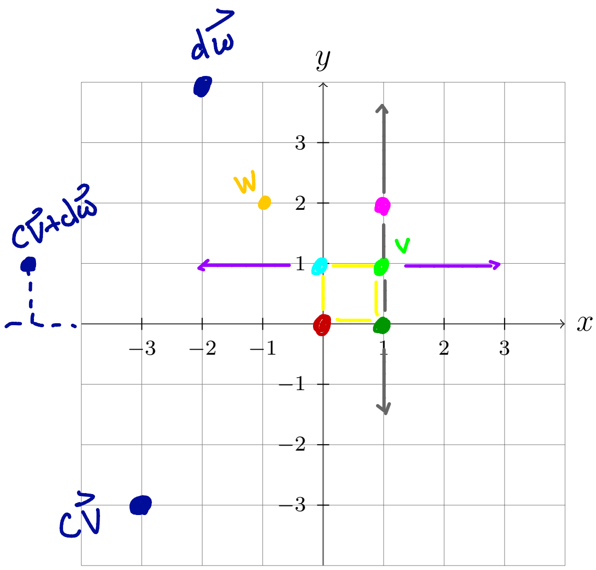
$$\overset{*}{=} \begin{bmatrix} .6 \\ .4 \end{bmatrix}$$

Matrix as a Transformation

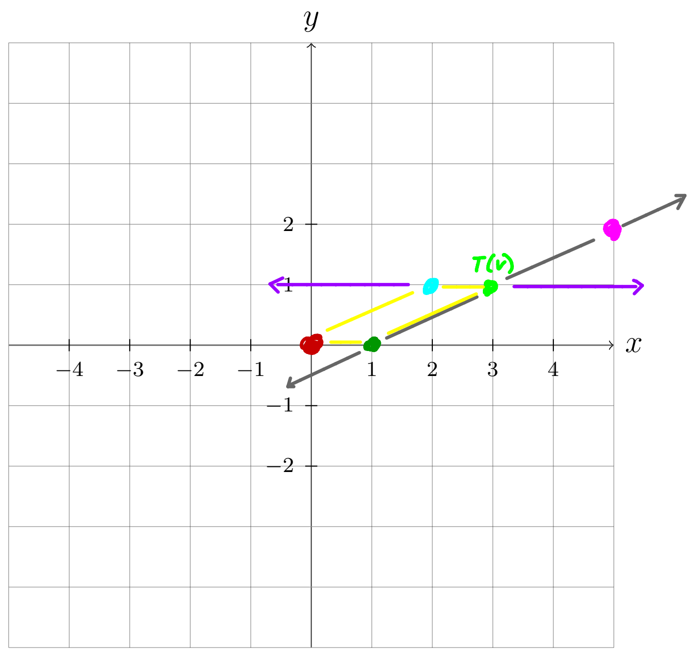
Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Problem 1. Plot the following points $p_1(0,0)$, $p_2(1,0)$, $p_4(0,1)$, $p_4(1,1)$, $p_5(1,2)$ in the domain and their images under T in the range.

Domain:



Range:



Problem 2. Let $c = -3$, $d = 2$, $\mathbf{v} = (1,1)$ and $\mathbf{w} = (-1,2)$.

(1) Calculate, plot and label in the domain the points/vectors $\mathbf{v} = (1,1)$, $\mathbf{w} = (-1,2)$ and

- $c\mathbf{v} = (-3,-3)$
- $d\mathbf{w} = (-2,4)$
- $c\mathbf{v} + d\mathbf{w} = (-5,1)$

(2) Calculate, plot and label in the range the points/vectors:

$T(\mathbf{v}) = (3,1)$

$T(\mathbf{w}) = (3,2)$

$cT(\mathbf{v}) = (-9,-3)$

$dT(\mathbf{w}) = (6,4)$

$cT(\mathbf{v}) + dT(\mathbf{w}) = (-3,1)$

$T(c\mathbf{v} + d\mathbf{w}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ Same!

You want to see that these are different calculations