(1) For each matrix A below, its columns will be labelled $\mathbf{a}_i$.

- Use R to write the columns of A as linear combinations of each other or state that this is not possible.
- Find a *basis* for $C(A)$, the column space of A and $N(A)$, the null space of A.
- Determine the *dimensions* of $C(A)$ and $N(A)$.

(a) $A \quad \rightarrow \quad R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

• Using the book algorithm for finding $N(A)$.
x_3 is free. So $x_1 + 2 = 0$
$x_2 + 3 = 0$

So $x_1 = -2, x_2 = 3, x_3 = 1$.
$S_3 = (-2, -3, 1)$
$N(A) = \{c\vec{S}_3 : c \text{ real}\}$
basis for $N(A) = \{\vec{S}_3\}$
dimension of $N(A) = 1$

• Just interpret $N(A)$. That is $A \cdot \vec{S}_3 = \vec{0}$ or

$$\begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ So $\underbrace{-2\vec{a}_1 - 3\vec{a}_2 + \vec{a}_3 = \vec{0}}$

Clearly, we must drop at least one of these three vectors in the basis of $C(A)$. How do we know 1 is sufficient?

basis for $C(A) = \{\vec{a}_1, \vec{a}_2\} = \{\vec{a}_1, \vec{a}_3\}$

$\dim(C(A)) = 2$

(b) $A \quad \rightarrow \quad R = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

$N(A)$

• Find special solutions
Set $x_3 = 1$, $x_5 = 0$.
$x_1 + 2 = 0$ or $x_1 = -2$ and $x_2 = x_4 = 0$
$S_3 = (-2, 1, 0, 0, 0)$
Set $x_5 = 1$, $x_3 = 0$.
$x_1 = 0$, $x_2 = -3$, $x_4 = 4$
$S_5 = (0, 0, -3, 4, 1)$
basis for $N(A) = \{\vec{S}_3, \vec{S}_5\}$, $\dim(N(A)) = 2$.

• $C(A)$.
$\vec{S}_3$ implies $-2\vec{a}_1 + \vec{a}_2 = \vec{0}$
$\vec{S}_5$ implies $-3\vec{a}_3 + 4\vec{a}_4 + \vec{a}_5 = \vec{0}$
basis for $C(A) = \{\vec{a}_1, \vec{a}_3, \vec{a}_4\}$
$= \{\vec{a}_2, \vec{a}_4, \vec{a}_5\}$
$\dim(C(A)) = 3$

(c) $A \quad \rightarrow \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$N(A) = \{\vec{0}\}$, basis $= \{\}$ $\dim(N(A)) = 0$
$\vec{a}_1, \vec{a}_2, \vec{a}_3$ are linearly independent.
basis for $C(A) = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \{\vec{i}, \vec{j}, \vec{k}\}$

(2) (summary of previous page) Suppose A is an $m \times n$ matrix of rank r.

(a) The dimension of $C(A)$ is <u>r</u> and a basis for $C(A)$ can be found by picking the columns of $\boxed{A}$ that have pivots in $\boxed{R}$.

(b) The dimension of $N(A)$ is <u>n−r</u> and a basis for $N(A)$ can be found by using the "free variable" algorithm

(3) Find a basis for the *row space of* A where $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 5 \\ 2 & -2 & -2 \end{bmatrix}$.

$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 2 & 5 & 2 \end{bmatrix}$

row space of A $= \left\{ c\begin{bmatrix}1\\0\\2\end{bmatrix} + d\begin{bmatrix}1\\1\\5\end{bmatrix} + e\begin{bmatrix}2\\-2\\-2\end{bmatrix};\ c,d,e \text{ real} \right\} = C(A^T)$

• <u>way 1</u>: $A^T \to \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. Using page 1, basis $C(A^T)$ $= \left\{ \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\5\end{bmatrix} \right\}$

pick of col 1, col 2 of A^T.

• <u>way 2</u>: From $\underline{\underline{R}}$ where $R = \text{rref}(A)$. ← No "T"

Do row ops to find R.

$$A = \begin{array}{c} \begin{matrix} 2 & 0 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 5 \\ 2 & -2 & -2 \end{bmatrix} \end{array} \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \xrightarrow{r_3 + 2r_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = R$$

$$A = \begin{bmatrix} \text{—} \ \vec{r}_1 \ \text{—} \\ \text{—} \ \vec{r}_2 \ \text{—} \\ \text{—} \ \vec{r}_3 \ \text{—} \end{bmatrix} \to \begin{bmatrix} \text{—} \ r_1 \ \text{—} \\ \text{—} \ r_2 - r_1 \ \text{—} \\ \text{—} \ r_3 - 2r_1 \ \text{—} \end{bmatrix} \to \begin{bmatrix} \text{—} \ r_1 \ \text{—} \\ \text{—} \ r_2 - r_1 \ \text{—} \\ \text{—} \ (r_3 - 2r_1) + 2(r_2 - 2r_1) \ \text{—} \end{bmatrix} = \begin{bmatrix} \text{—} \ r_1 \ \text{—} \\ \text{—} \ r_2 - r_1 \ \text{—} \\ \text{—} \ r_3 - 6r_1 + 2r_2 \ \text{—} \end{bmatrix} = R$$

!! $\begin{pmatrix} \text{row space} \\ \text{of} \\ A \end{pmatrix} = \begin{pmatrix} \text{row space} \\ R \end{pmatrix}$ basis for $C(A^T)$ $= \left\{ \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix} \right\}$

Revisit 1b. basis row space $= \left\{ \begin{bmatrix}1\\0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\-4\end{bmatrix} \right\}$ dim $= 3$

So $\dim\begin{pmatrix}\text{row space}\\ \text{of } A\end{pmatrix} = \dim\left(C\left(A^T\right)\right) = r = \text{rank of } A.$

That is, the dimension of the col. space and row space are the same even though they may not even be in the same space!!
If A is $m \times n$ matrix, $C(A)$ in $\mathbb{R}^m$
row space in $\mathbb{R}^n$.

Last topic: "left" null space of A $\equiv$ null space of A^T

$\boxed{A^T \vec{y} = \vec{0}}$ is equivalent to $\boxed{\vec{y}^T A = \vec{0}}$

(using $B\vec{x} = \vec{x}^T B^T$)

If $m \times n$ matrix A has rank r, then

$$\begin{matrix}\text{dimension of}\\ N(A^T)\end{matrix} = \begin{matrix}\text{dimension of}\\ \text{left null space}\\ \text{of } A\end{matrix} = \boxed{m - r}$$

thinking: $C(A^T) = r$. So $N(A^T) = \#\text{cols } A^T - r$
$= m - r$

Where do these spaces live supposing A is $m \times n$ ...

$$A = \begin{array}{c|c} & \begin{matrix}1 & 2 & \cdots & n\end{matrix} \\ \begin{matrix}1\\2\\ \vdots\\ m\end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}\end{array}$$

$\mathbb{R}^m$			$\mathbb{R}^n$
$r = C(A)$	dimensions sum to m	dims sum to n	row space A $= C\left(A^T\right) = r$
$N(A^T)$			$N(A)$

$r = \text{rank}(A)$