(1) Find a basis for the four subspaces of A.

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & -2 \\ 1 & 2 & -2 & 0 & 0 \\ 1 & 2 & -1 & 1 & -4 \\ 2 & 4 & 3 & -3 & 2 \end{bmatrix} \quad \rightarrow \quad R = \begin{array}{ccccc} & x_2 & & & x_5 \\ \end{array} \begin{bmatrix} 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

rank $A = 3$

$$A^T \quad \rightarrow \quad R = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A\vec{x} = \vec{b}$

5 vector ($\vec{x}$), 4-vector ($\vec{b}$)

$\mathbb{R}^5$

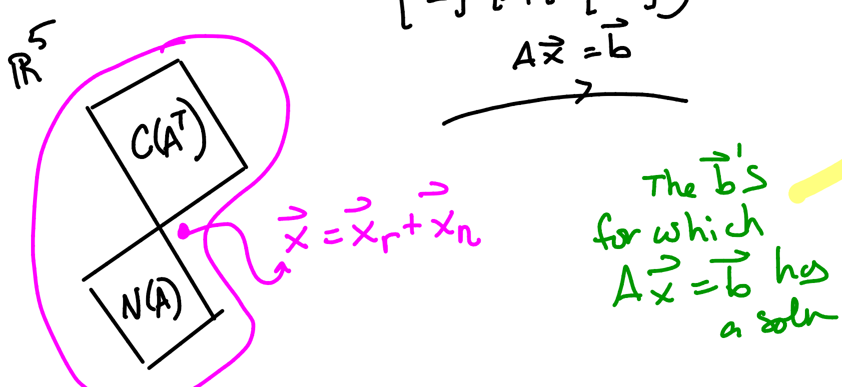
$$N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

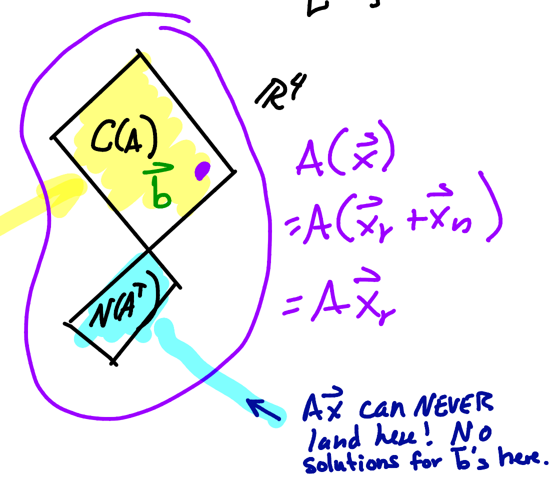
$\mathbb{R}^4$

$$C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

$$N(A^T) = \text{span}\left\{ \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

row space of A $= C(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -3 \end{bmatrix} \right\}$





(2) Why are the principles below true?

(a) If v and w are orthogonal, then $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

- Pyth. Thm. or
- $\|\vec{v} + \vec{w}\|^2 = (v+w)\cdot(v+w) = v\cdot v + \underbrace{2v\cdot w}_{=0} + w\cdot w = \|v\|^2 + \|w\|^2$!

(b) If v and w are orthogonal, then v and w are linearly independent.

NONZERO and

If $\vec{v} = k\vec{w}$, then $v \cdot w = (kw)\cdot w = k\|w\|^2 > 0$

(c) For any matrix A, $C(A^T) \perp N(A)$.

- Observation: If $\vec{v} = \text{row}_i(A)$ and $\vec{w}$ in $N(A)$, then $\vec{v} \cdot \vec{w} = 0$ b/c

$$A\vec{w} = \begin{bmatrix} \text{row}_1 \cdot \vec{w} \\ \text{row}_2 \cdot \vec{w} \\ \vdots \\ \text{row}_k \cdot \vec{w} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- It follows that $\vec{w}$ is orthog to every linear combo of rows of A.
(b/c $(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) \cdot \vec{w} = c_1\underbrace{(v_1 \cdot w)}_{0} + c_2\underbrace{(v_2 \cdot w)}_{0} + \cdots + c_k\underbrace{(v_k \cdot w)}_{0} = 0$

- It follows that every linear combo of rows is orthog to every linear combo of vectors in $N(A)$. (Same argument.)

(d) For any matrix A, $C(A) \perp N(A^T)$.

Call $B = A^T$ and apply (c) to $C(B) \perp N(B^T)$.

(e) If vector space V has two bases $B_1 = \{v_1, v_2, \cdots, v_m\}$ and $B_2 = \{w_1, w_2, \cdots, w_n\}$, then $m = n$. (That is, the notion of the dimension of a vector space is well-defined.) (Sppse $m < n$)

Since B_1 is a basis $c_1v_1 + c_2v_2 + \cdots + c_mv_m = w_1$. (Same for $w_2, w_3, \ldots, w_n$)

Let $C = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & & | \end{bmatrix}}_{m \times n}$ where $\vec{c}_1 = (c_1, c_2, \ldots, c_m)$. Since $m < n$, C is wide. So rref(C) has a free variable. So $N(C)$ has nonzero vectors. So w_i's are NOT linearly independent. That is a solution $\vec{x}$ to $C\vec{x} = \vec{0}$, gives a nontrivial linear combo of $\vec{w}$'s that equal zero.

(f) If the dimension of a vector space, V, is d, then you can tell that a set of d vectors is a basis of V by checking whether it is...

① linearly indep OR ② spans V

(In general, you have to check both)