(1) Find the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and associated eigenvectors, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ for the matrix $\mathbf{A}$ below.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad A - \lambda I_3 = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_3) = (3-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)(1-\lambda) = 0$$

$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

Find x_1: $A - 1 \cdot I_3 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So $\vec{x}_1 = (0,0,1)$

Find x_2: $A - 2I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ So $\vec{x}_2 = (1,0,0)$

Find x_3: $A - 3I_3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$. So $-x_1 + x_2 = 0$

$x_2 - 2x_3 = 0$

Pick $x_3 = 1$. So $x_2 = 2$ and, then $x_1 = 2$.

So $\vec{x}_3 = (2,2,1)$.

(2) Define the two matrices $\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$ and $\boldsymbol{\Lambda}$ to be a 3 by 3 *diagonal* matrix with $\lambda_1, \lambda_2, \lambda_3$ down the main diagonal *in that order* and write them below.

$$X = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Is $\mathbf{X}$ invertible or not? Why?

X is invertible.

The columns are linearly independent.

(b) Compute $\mathbf{AX}$.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 6 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{x}_1 & 2\vec{x}_2 & 3\vec{x}_3 \\ | & | & | \end{bmatrix}$$

(c) Compute $\mathbf{X\Lambda}$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 6 \\ 1 & 0 & 3 \end{bmatrix}$$

the same!

(d) Explain, without computation, that $\mathbf{A} = \mathbf{X\Lambda X}^{-1}$

Since $AX = X\Lambda$ and X^{-1} exists, $AXX^{-1} = X\Lambda X^{-1}$.

So $A = X\Lambda X^{-1}$

(e) Use your answer from part (d), to find another way to compute $\mathbf{A}^5$. Are there advantages to the alternative approach?

$$A^5 = (X\Lambda \underbrace{X^{-1})(X}_{I_3} \Lambda \underbrace{X^{-1})(X}_{I_3} \Lambda \underbrace{X^{-1})(X}_{I_3} \Lambda \underbrace{X^{-1})(X}_{I_3} \Lambda X^{-1})$$

$= X\Lambda^5 X^{-1}$. Observe $\Lambda^5 = \begin{bmatrix} 1^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 3^5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{bmatrix}$