Final Exam Questions

1. How to compute $\|v\|$ and what does it mean?

2. How do you find the angle between two vectors? How do you know if two vectors are orthogonal (perpendicular) or at an acute (or obtuse) angle.

3. Assume you are given a collection of vectors, say v_1, v_2, v_3, how can you *show* they are linearly independent? Linearly dependent?

4. Assume you are given a collection of vectors, say v_1, v_2, v_3, how can you show that they form a basis? That they do not form a basis? That they are an orthonormal basis?

5. Assume a set of vectors is a basis, say v_1, v_2, v_3, and you are given another vector, say v, how do you write v vector as a linear combination of v_1, v_2, v_3? What are the advantages of an orthonormal set of vectors?

6. Given vectors a_1, a_2, a_3, how do you implement the Gram-Schmidt algorithm? What properties do the q's have? How are those properties related to the a's?

7. If q_1, q_2 and q_3 are orthonormal in $\mathbb{R}^3$, explain how you know they form a basis? Since they do, given $x \in \mathbb{R}^3$ we can write $x = \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3$ for some numbers α_1, α_2 and α_3. What are the numbers α_k? Hint: they can be expressed using inner products: equation (5.5) in the text.

8. Can you find the matrix A such that Ax is the transformation of the plane that rotates x by 60 degrees and then reflects it about the y axis?

9. Given a vector a and a vector b, how do you compute the convolution $a * b$? What is the matrix T such that $a * b = Tb$?

10. Know how to represent a linear function in terms of a matrix.
For example, suppose f is the function that takes (x_1, x_2, x_3) to $((x_2 + x_3)/2, -x_3, x_1)$. What is its representation in terms of a matrix?

11. Similarly, suppose f is a linear map from R^2 to R^4 and $f(e_1) = (-1, 3, 4, 3)$ and $f(e_2) = (2, 3, 4, 9)$. What is the representation of f in terms of a matrix?

12. How can you show that a function is linear? Affine? Not affine? Think up your own examples of functions from R^3 to R^3 that are linear, affine, and neither.

13. If asked for the coefficients of a quadratic polynomial p with $p(x_i) = y_i$ for $i = 1, \ldots, 5$, can you set up a system of linear equations to solve for the coefficients?

14. How do you solve $Ax = b$ if A is lower triangular? What if A is upper triangular?

15. How do you check that a matrix A is orthogonal?

16. How do you check that the *columns* of the matrix A form an orthogonal set of vectors? How is this question different from the previous one?

17. What is the column perspective of matrix-matrix multiplication? What is the row perspective?

18. Express the following task as a matrix algebra task: "Find a linear combination of vectors a_1, a_2 and a_3 that equals b". This gets at the column interpretation of matrix-vector multiplication (page 119).

19. What is the transpose of a matrix product?

20. If you multiply a 3×3 matrix A on the left by $\mathrm{diag}(1, 2, 3)$ what is the result? How about if you multiply A on the right by $\mathrm{diag}(1, 2, 3)$?

21. Find a 4×5 matrix L such that when you multiply any $5 \times k$ matrix A on the left by L, the result is the matrix A with its bottom row removed.

22. Given a matrix A with linearly independent columns, how do you compute its QR factorization? This is related to the Gram-Schmidt algorithm, and you should review how you convert the steps of the Gram-Schmidt algorithm into the entries of the matrices of the QR factorization. **You will be asked to show that you know how to do this.** See also homework 8, additional problem 2.

23. Now, given the QR factorization of a square matrix A, how do you solve $Ax = b$? This is a two step procedure. If I give you Q and R, can you carry out the steps?

24. Suppose A has a left inverse. Show that the columns of A have to be linearly independent.

25. Once you know that a square matrix has linearly independent columns you know a whole bunch of other things are true. Name as many as you can. How many solutions of $Ax = 0$ are there? Why?

26. Similarly, once you know that a square matrix is **not** invertible (or is singular), you know a whole bunch of things. Name as many as you can. How many solutions to $Ax = 0$ are there?

27. What is the inverse of a matrix product of invertible matrices? What is the inverse of the transpose of an invertible matrix?

28. Suppose A is an 3×3 matrix. How do you find A^{-1}? If you have a QR factorization of A, how do you find A^{-1}?

29. How to you find the pseudoinverse of a matrix A?

30. How can you use the pseudoinverse of a matrix A?

31. How do you find *the least squares approximate solution* of $Ax = b$, namely $\hat{x}$, and how do your interpret it?

32. What are the *normal equations* are for a system $Ax = b$.

33. If A has a QR-factorization, what is it's pseudoinverse?

34. Suppose A is tall and $A = QR$, where Q has orthonormal columns and R is upper-triangular, How do your compute the pseudo inverse $A^{\dagger}$? How do you find the least squares approximate solution to $Ax = b$?

35. Translate a problem – such as approximating a given set of points with a polynomial – to a least squares approximation problem. Set up a matrix system to solve this problem and solve it. (See HW 10 #3.)

36. Suppose $\hat{x}$ is the least squares approximate solution to $Ax = b$ and suppose y is any other n-vector. What can you conclude about $\hat{x}$ and y?

37. You should know what the null space of a matrix A is, how to find it, and how it relates to linear functions and matrix invertibility.

38. Suppose you have two vectors $w_1 = (1, 2, 3, 4)$ and $w_2 = (2, 3, 4, 5)$,. Set up a matrix A such that the null space of A is the set of all vectors perpendicular to both w_1 and w_2. Then determine the set of vectors in this null space.

39. Suppose A is a wide matrix and w is a solution to $Ax = b$, how do you find *all* solutions to $Ax = b$?
40. How do your find the determinant of a matrix, including very large but sparse ones.
41. You should know basic ”determinant algebra” including
    - how elementary row operations and matrix operation ($+$, $*$, inverses) affect the determinant
    - how the determinant relates to the null space and, thus, invertibility
42. You should know what eigenvalues and eigenvectors of a matrix A are, how to find them, how to confirm them, and what they indicate about A as a linear function.
43. What are the eigenvalues of an upper triangular matrix?
44. Given a square matrix A and vector v, how would you check if v was an eigenvector of A?
45. If x is an eigenvector of A with eigenvalue λ, what can you say about $A^k x$ for a positive integer k?