Final Exam Review

The Basics

The Final Exam will cover Chapters 1-3,5-8,10-12, null spaces, determinants, eigenvalues and eigenvectors. It will be weighted toward topics at the end of the semester. You will have 2 hours to take the exam. You can bring in 1 page of notes with writing on one side. You can bring a calculator. You cannot use your cell phone or a computer. You will need to show your work.

Chapter-by-Chapter Review of Topics

- Ch 1

    - Know the language and notation of vectors and how to perform basic vector operations including rules of vector algebra.
    - Know the terminology of *linear combinations of vectors*, *inner product of vectors*, and *unit vectors*.
    - Recall familiar applications.

- Ch 2

    - Know the definition of a linear function and the basic strategies for determining if a function is or is not linear. Specifically, be able to show that a function is linear by exhibiting it in the form of ab inner product: $f(x) = a^T x$ and be able to show that a function is not linear by finding a counter-example.
    - Know the definition of an affine function and the basic strategy for determining if a function is or is not affine.
    - Review familiar applications and interpretations of linear/affine functions.

- Ch 3

    - Know how to compute the norm of a vector, the distance between two vectors, and the angle between two vectors.
    - Know how to tell if two vectors are orthogonal (perpendicular) or at an acute (or obtuse) angle.
    - Know the triangle inequality and the Cauchy-Schwartz inequality.
    - Be able to do algebraic manipulations involving norms and inner products

- Ch 5

    - Know the definition of linear independence and linear dependence and how to use them to *show* that a set of vectors is or is not linearly independent (dependent).
    - Know the **independence-dimension inequality** or **Fact A**.
    - Know the definition of a basis and an orthonormal basis. Understand how to determine if a set of vectors is a basis and how to write one vector as a linear combination of others. What are the advantages of an orthonormal set of vectors?
    - Know the Gram-Schmidt algorithm. Be able to implement it in a simple case (turn a_1, a_2, a_3 into q_1, q_2, q_3 with intermediate vectors $\tilde{q}_k$.) What properties do the q's have? How are those properties related to the a's?

- If q_1, q_2 and q_3 are orthonormal in $\mathbb{R}^3$, explain how you know they form a basis. Since they do, given $x \in \mathbb{R}^3$ we can write $x = \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3$ for some numbers α_1, α_2 and α_3. What are the numbers α_k? Hint: they can be expressed using inner products: equation (5.5) in the text.

- Ch 6

  - Know how to reference matrices and to do basic calculations including matrix-vector multiplication.

Ch 7 Matrix Examples

This chapter introduced a variety of applications of matrices and of the matrix-vector product. These include transformations of the plane or 3-space (like rotation by θ or reflection about a line), selector matrices, incidence matrix of a graph, and convolution. Recall that it also included a strategy: build the matrix of a linear transformation/operation by finding the image of the e_i's under the transformation.

- Can you interpret the matrix-vector product in context?

- Can you find the matrix A such that Ax is the transformation of the plane that rotates x by 60 degrees and then reflects it about the y axis?

- We've seen a lot of connections between linear algebra and calculus questions. For example, how do you represent derivatives of cubic polynomials in terms of matrix multiplication? How do you represent antiderivatives of cubic polynomials in terms of matrix multiplication?

- Given a vector a and a vector b, how do you compute the convolution $a * b$? What is the matrix T such that $a * b = Tb$?

Ch 8 Linear Equations

This chapter introduces the notion of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ with a focus on linear and affine functions.

- Know how to represent a linear function in terms of a matrix.
  For example, suppose f is the function that takes (x_1, x_2, x_3) to $(x_2, -x_3, x_1)$. What is its representation in terms of a matrix?

- Similarly, suppose f is a linear map from R^2 to R^4 and $f(e_1) = (-1, 3, 4, 3)$ and $f(e_2) = (2, 3, 4, 9)$. What is the representation of f in terms of a matrix?

- How can you show that a function is linear? affine? Not affine?

- If asked for the coefficients of a quadratic polynomial p with $p(x_i) = y_i$ for $i = 1, \ldots, 5$, can you set up a system of linear equations to solve for the coefficients?

- How do you solve $Ax = b$ if A is lower triangular? What if A is upper triangular?

Ch 10 Matrix Multiplication

This chapter introduced matrix-matrix multiplication and finished with QR factorization. It included the algebra of matrix-matrix multiplication (see page 179) and orthogonal matrices.

- How do you check that a matrix A is orthogonal? Restate your answer to the question using the language of Chapter 11.

- What is the column perspective of matrix-matrix multiplication? What is the row perspective?

- Express the following task as a matrix algebra task: "Find a linear combination of vectors a_1, a_2 and a_3 that equals b". This gets at the column interpretation of matrix-vector multiplication (page 119).

- What is the transpose of a matrix product?

- If you multiply a 3×3 matrix A on the left by $\text{diag}(1, 2, 3)$ what is the result? How about if you multiply A on the right by $\text{diag}(1, 2, 3)$?

- Find a 4×5 matrix L such that when you multiply any $5 \times k$ matrix A on the left by L, the result is the matrix A with its bottom row removed.

- Given a matrix A with linearly independent columns, how do you compute its QR factorization? This is related to the Gram-Schmidt algorithm, and you should review how you convert the steps of the Gram-Schmidt algorithm into the entries of the matrices of the QR factorization. **You will be asked to show that you know how to do this.** See also homework 8, additional problem 2.

- Now, given the QR factorization of a square matrix A, how do you solve $Ax = b$? This is a two step procedure. If I give you Q and R, can you carry out the steps?

Ch 11 Matrix Inverses

This chapter introduces the idea of left/right/both-sided inverses and encourages a multifaceted view - including row-focused and column-focused views. We discussed invertibility conditions, applications of inverses to solutions to systems of equations, and how multiplicative inverses interact with other matrix operations such as multiplication and transposes.

- Suppose A has a left inverse. Show that the columns of A have to be linearly independent.

- Once you know that a square matrix has linearly independent columns you know a whole bunch of other things are true. Name as many as you can. How many solutions of $Ax = 0$ are there? Why?

- Similarly, once you know that a square matrix in **not** invertible (or is singular), you know a whole bunch of things. Name as many as you can. How many solutions to $Ax = 0$ are there?

- What is the inverse of a matrix product of invertible matrices? What is the inverse of the transpose of an invertible matrix?

- Suppose A is an 3×3 matrix. What matrix equation do you solve to determine the first column of A^{-1}? What equation to you solve to find the third column of A^{-1}? If you have a QR factorization of A, can you determine these columns?

Ch 12 Least Squares Solutions

- Here is where we first learned about and used the **pseudoinverse** of a matrix A (which is technically in Chapter 11). Recall that the pseudoinverse of A is denoted $A^\dagger$.

- You should understand what is meant by *the least squares approximate solution* of $Ax = b$ and the meaning of $\hat{x}$.

- You should know what the *normal equations* are for a system $Ax = b$.

- You should know how the pseudoinverse, QR-factorization, and solving least squares problems are related.

- Suppose A is tall and $A = QR$, where Q has orthonormal columns and R is upper-triangular, How do your compute the pseudo inverse $A^\dagger$? How do you find the least squares approximate solution to $Ax = b$?

- Translate a problem – such as approximating a given set of points with a polynomial – to a least squares approximation problem. Set up a matrix system to solve this problem and solve it. (See HW 10 #3.)

- Suppose $\hat{x}$ is the least squares approximate solution to $Ax = b$ and suppose y is any other n-vector. What can you conclude about $\hat{x}$ and y?

Last Topics Null Spaces, Determinants, Eigenvalues and Eigenvectors

- You should know what the null space of a matrix A is, how to find it, and how it relates to linear functions and matrix invertibility.

- Suppose you have two vectors $w_1 = (1, 2, 3, 4)$ and $w_2 = (2, 3, 4, 5)$,. Set up a matrix A such that the null space of A is the set of all vectors perpendicular to both w_1 and w_2. Then determine the set of vectors in this null space.

- Suppose A is a wide matrix and w is a solution to $Ax = b$, how do you find *all* solutions to $Ax = b$?

- You should know how to find the determinant of a matrix, including very large but sparse ones.

- You should know basic "determinant algebra" including

  - how elementary row operations and matrix operation $(+, *, \text{inverses})$ affect the determinant
  - how the determinant relates to the null space and, thus, invertibility

- You should know what eigenvalues and eigenvectors of a matrix A are, how to find them, how to confirm them, and what they indicate about A as a linear function.

- What are the eigenvalues of an upper triangular matrix?

- Given a square matrix A and vector v, how would you check if v was an eigenvector of A?

- If x is an eigenvector of A with eigenvalue λ, what can you say about $A^k x$ for a positive integer k?