Fall 2024 *Math F314X*

Linear Algebra: Final Exam

Name: Solutions

Rules:

- Show your work.
- You may have a single handwritten sheet of paper with writing on one side.
- You may use a calculator

Problem	Possible	Score
1	16	
2	20	
3	20	
4	15	
5	11	
6	10	
7	8	
Total	100	

1. (16 points)

(a) **Demonstrate** that the vectors $a_1 = \begin{bmatrix}2\\0\\0\end{bmatrix}, a_2 = \begin{bmatrix}0\\1\\2\end{bmatrix}, a_3 = \begin{bmatrix}0\\-1\\1\end{bmatrix}$ are linearly independent.

This should involve both calculations and an explanation of why that calculation allows one to conclude the vectors are linearly independent.

option 1: $A = \begin{bmatrix}2&0&0\\0&1&-1\\0&2&1\end{bmatrix}$; $\det(A) = 2\begin{vmatrix}1&-1\\2&1\end{vmatrix} = 2(1-(-2)) = 6 \neq 0$

Since $\det(A) \neq 0$, A is invertible. So its columns are linearly independent.

option 2: Look for solutions to $\boxed{\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0}$ and show there is only 1 solution: $\beta_1 = \beta_2 = \beta_3 = 0$.

So solve $\underbrace{\begin{bmatrix}2&0&0\\0&1&-1\\0&2&1\end{bmatrix}}_{A=}\vec{\beta} = \vec{0}$. So $\begin{bmatrix}2&0&0\\0&1&-1\\0&2&1\end{bmatrix} \xrightarrow[r_3-2r_2\to r_3]{\frac{1}{2}r_1\to r_1} \begin{bmatrix}1&0&0\\0&1&-1\\0&0&3\end{bmatrix} \xrightarrow[r_2+r_3\to r_2]{\frac{1}{3}r_3\to r_3} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Since $\mathrm{rref}(A) = I_3$, $\vec{\beta} = \vec{0}$ is the unique solution.

(b) **Demonstrate** the vectors $v_1 = \begin{bmatrix}2\\4\\0\\0\end{bmatrix}, v_2 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}, v_3 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, v_4 = \begin{bmatrix}1\\4\\-2\\0\end{bmatrix}$ are linearly dependent by writing one vector as a linear combination of the others.

$\frac{1}{2}v_1 + 2v_2 + 0 \cdot v_3 = v_4$ ← by inspection.

systematically:

$$\begin{bmatrix}2&0&0&1\\4&1&1&4\\0&-1&1&-2\\0&0&1&0\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix};\quad \begin{bmatrix}2&0&0&1\\4&1&1&4\\0&-1&1&-2\\0&0&1&0\end{bmatrix} \xrightarrow{r_2-2r_1\to r_2} \begin{bmatrix}2&0&0&1\\0&1&1&2\\0&-1&1&-2\\0&0&1&0\end{bmatrix} \xrightarrow{r_3+r_2\to r_3}$$

$$\begin{bmatrix}2&0&0&1\\0&1&1&2\\0&0&2&0\\0&0&1&0\end{bmatrix} \xrightarrow[r_3-2r_4\to r_3]{r_2-r_4\to r_2} \begin{array}{c}\begin{bmatrix}2&0&0&1\\0&1&0&2\\0&0&0&0\\0&0&1&0\end{bmatrix}\\ \begin{array}{cccc}\beta_1&\beta_2&\beta_3&\beta_4\end{array}\end{array};$$

$2\beta_1 + \beta_4 = 0$
$\beta_2 + 2\beta_4 = 0$
$\beta_3 = 0$

If $\beta_4 = -1$, then $\beta_1 = +\frac{1}{2}$ and $\beta_2 = 2$.

2. (20 points) Let $\mathcal{S}$ be the system of equations: $\boxed{\begin{array}{ccccc} x_1 & & & = & 1 \\ & & x_2 & = & 1 \\ x_1 & + & x_2 & = & 1 \end{array}}$. Observe that this system has no exact solution.

(a) Write this system in the matrix form $Ax = b$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) Find $A^T A$.

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(c) Find $(A^T A)^{-1}$

$$(A^T A)^{-1} = \frac{1}{2 \cdot 2 - 1 \cdot 1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

(d) Find $A^\dagger$, the pseudoinverse of A.

$$A^+ = (A^T A)^{-1} A^T = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \end{bmatrix}$$

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$$\begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \end{bmatrix}$$

For reference, S is:
$$\boxed{\begin{array}{ccccc} x_1 & & & = & 1 \\ & & x_2 & = & 1 \\ x_1 & + & x_2 & = & 1 \end{array}}.$$

(e) Find $\hat{x}$, the least squares approximate solution to the system S.

$$\hat{x} = A^{+} b = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$$

(f) Suppose someone chooses their approximate solution to S to be $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

i. Explain (in words and/or correct mathematical notation) why $\hat{x}$ is a *better* approximate solution than z.

(words) $A\hat{x}$ will be as close to b as is possible; Az can only be farther away from b.

(symbols) $\|A\hat{x} - b\| \leq \|Az - b\|$

ii. Complete the calculation that demonstrates your description above is correct.

$$A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ 4/3 \end{bmatrix}; \quad \|A\hat{x} - b\| = \|(-\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3})\| = \sqrt{\tfrac{1}{9} + \tfrac{1}{9} + \tfrac{1}{9}} = \frac{1}{\sqrt{3}}$$

$$Az = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}; \quad \|Az - b\| = \|(0, 0, 1)\| = 1$$

We observe: $\frac{1}{\sqrt{3}} < 1$ ☺

3. (20 points) Let $a_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, a_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}, a_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$. The first parts of this problem will ask you to go through part of the Gram-Schmidt algorithm.

(a) Find q_1, the first vector obtained via Gram-Schmidt.

$$\bar{q}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$$

(b) It is a fact that $\overline{q_2} = \frac{1}{2}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}$ and $q_2 = \begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$. Find q_3.

$$\bar{q}_3 = a_3 - \left(\frac{a_3^T \bar{q}_2}{\|\bar{q}_2\|^2}\right)\bar{q}_2 - \left(\frac{a_3^T \bar{q}_1}{\|\bar{q}_1\|^2}\right)\bar{q}_1$$

- $a_3^T \bar{q}_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}\begin{bmatrix}1/2\\0\\-1/2\end{bmatrix} = 1$

$$= \begin{bmatrix}2\\1\\0\end{bmatrix} - \frac{1}{1/2}\begin{bmatrix}1/2\\0\\-1/2\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

- $\|\bar{q}_2\|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

- $a_3^T \bar{q}_1 = \begin{bmatrix}2\\1\\0\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = 2$

$$= \begin{bmatrix}2\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\-1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

- $\|\bar{q}_1\|^2 = 1 + 1 = 2$

So $\bar{q}_3 = q_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

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(c) Let $A = [a_1\ a_2\ a_3] = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. (Note that the $a_i's$ on this page are the same as on the provious page.

i. Determine the Q, in the QR-factorization of A.

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

ii. Find the *second row* of R in the QR-factorization of A. That is, you should find R_{21}, R_{22}, and R_{23}.

$$R_{21} = 0$$

$$R_{22} = a_2^T q_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = 1/\sqrt{2}$$

$$R_{23} = a_3^T q_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \sqrt{2}$$

iii. Is A invertible? Justify your conclusion.

Yes. The columns of A are linearly independent.

4. (15 points) Let $C = \begin{bmatrix} 6 & 0 & 0 \\ 1 & 2 & 4 \\ -1 & 2 & 0 \end{bmatrix}$. You must show your work to earn full points.

(a) Find all eigenvalues of the matrix C.

$$\begin{vmatrix} 6-\lambda & 0 & 0 \\ 1 & 2-\lambda & 4 \\ -1 & 2 & -\lambda \end{vmatrix} = (6-\lambda)\big((2-\lambda)(-\lambda) - 4(2)\big)$$

$$= (6-\lambda)(\lambda^2 - 2\lambda - 8)$$

$$= (6-\lambda)(\lambda-4)(\lambda+2)$$

eigenvalues: $\lambda = 6, 4, -2$

(b) For the largest eigenvalue of C, find an associated eigenvector.

$\lambda = 6$: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & -4 & 4 \\ -1 & 2 & -6 \end{bmatrix} \xrightarrow[\text{rotate rows}]{r_3 + r_2 \to r_3} \begin{bmatrix} 1 & -4 & 4 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2} r_2 \to r_2} \begin{bmatrix} 1 & -4 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{4r_2 + r_1 \to r_1}$ $\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

$x_1 + 8x_3 = 0$

$x_2 + x_3 = 0$

Choose $x_3 = -1$: $v = \begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix}$

(c) Suppose that v is the eigenvector you found in part (b) above. Determine $C^{10}v$.

$6^{10} v$.

5. (11 points)

(a) Suppose M is an **orthogonal** $n \times n$ matrix.

i. Can we draw any conclusions about the *null space of* M? Explain and justify.

The null space is $\{\vec{0}\}$.

Orthogonal matrices are invertible. Thus, the system $Ax=0$ has a unique solution.

ii. Can we draw any conclusion about whether the *rows* of M are linearly independent? Explain and justify.

The rows are linearly independent because Q is invertible and, thus, has a right inverse. Thus, its rows are linearly independent.

(b) The matrix $M = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$ is orthogonal. Write the vector $v = (1, 2, 0)$ as a linear combination of the columns of M.

Let $a_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$

$v^T a_1 = 4/3$

$v^T a_2 = 2/3$

$v^T a_3 = 5/3$

$v = (v^T a_1) a_1 + (v^T a_2) a_2 + (v^T a_3) a_3$

$= \frac{4}{3} a_1 + \frac{2}{3} a_2 + \frac{5}{3} a_3$

6. (10 points) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \\ 4 & 2 \end{bmatrix}$

(a) Explain why A cannot have a right inverse.

The rows of A are linearly dependent.

(b) Find a left inverse of A.

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/5 & 0 \end{bmatrix}$$

(c) Is your answer in part (b) unique? Justify your conclusion.

No

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/10 & 1/4 \end{bmatrix}$$

7. (8 points) Determine whether each function below is a linear function $f : \mathbb{R}^2 \to \mathbb{R}^3$. If f is linear, show this by writing $f(x) = Ax$ for an appropriate matrix A. If f is not linear, find particular vectors and scalars for which f fails to be linear.

(a) $f(x_1, x_2) = (\frac{2x_1 - x_2}{2}, \frac{-x_1 + 2x_2}{2}, \frac{x_1 + x_2}{2})$

linear

$$A = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

(b) $f(x_1, x_2) = (1 + x_1, 2 + x_2, 0)$

not linear

choose $\alpha = 2, \beta = 0, u = (1,1), v = (0,0)$

$f(\alpha u + \beta v) = f(\alpha u) = f(2,2) = (3,4)$ not equal

$\alpha f(u) + \beta f(v) = \alpha f(u) = 2f(1,1) = 2(2,3) = (4,6)$