Fall 2024 *Math F314X*

Linear Algebra: Final Exam

Name: ____________________

Rules:

- Show your work.
- You may have a single handwritten sheet of paper with writing on one side.
- You may use a calculator

Problem	Possible	Score
1	16	
2	20	
3	20	
4	15	
5	11	
6	10	
7	8	
Total	100	

1. (16 points)

(a) **Demonstrate** that the vectors $a_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, a_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ are linearly independent. This should involve both calculations and an explanation of why that calculation allows one to conclude the vectors are linearly independent.

(b) **Demonstrate** the vectors $v_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 0 \end{bmatrix}$ are linearly dependent by writing one vector as a linear combination of the others.

2. (20 points) Let $\mathcal{S}$ be the system of equations: $\boxed{\begin{array}{ccccc} x_1 & & & = & 1 \\ & & x_2 & = & 1 \\ x_1 & + & x_2 & = & 1 \end{array}}$. Observe that this system has no exact solution.

(a) Write this system in the matrix form $Ax = b$.

(b) Find $A^T A$.

(c) Find $(A^T A)^{-1}$

(d) Find $A^\dagger$, the pseudoinverse of A.

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For reference, S is:
$$\boxed{\begin{array}{ccccc} x_1 & & & = & 1 \\ & & x_2 & = & 1 \\ x_1 & + & x_2 & = & 1 \end{array}}.$$

(e) Find $\hat{x}$, the least squares approximate solution to the system S.

(f) Suppose someone chooses their approximate solution to S to be $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

i. Explain (in words and/or correct mathematical notation) why $\hat{x}$ is a *better* approximate solution than z.

ii. Complete the calculation that demonstrates your description above is correct.

3. (20 points) Let $a_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, a_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}, a_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$. The first parts of this problem will ask you to go through part of the Gram-Schmidt algorithm.

   (a) Find q_1, the first vector obtained via Gram-Schmidt.

   (b) It is a fact that $\overline{q_2} = \frac{1}{2}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}$ and $q_2 = \begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$. Find q_3.

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(c) Let $A = [a_1 \; a_2 \; a_3] = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. (Note that the $a_i's$ on this page are the same as on the provious page.

i. Determine the Q, in the QR-factorization of A.

ii. Find the *second row* of R in the QR-factorization of A. That is, you should find R_{21}, R_{22}, and R_{23}.

iii. Is A invertible? Justify your conclusion.

4. (15 points) Let $C = \begin{bmatrix} 6 & 0 & 0 \\ 1 & 2 & 4 \\ -1 & 2 & 0 \end{bmatrix}$. You must show your work to earn full points.

(a) Find all eigenvalues of the matrix C.

(b) For the largest eigenvalue of C, find an associated eigenvector.

(c) Suppose that v is the eigenvector you found in part (b) above. Determine $C^{10}v$.

5. (11 points)

(a) Suppose M is an **orthogonal** $n \times n$ matrix.

i. Can we draw any conclusions about the *null space of* M? Explain and justify.

ii. Can we draw any conclusion about whether the *rows* of M are linearly independent? Explain and justify.

(b) The matrix $M = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$ is orthogonal. Write the vector $v = (1, 2, 0)$ as a linear combination of the columns of M.

6. (10 points) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \\ 4 & 2 \end{bmatrix}$

(a) Explain why A cannot have a right inverse.

(b) Find a left inverse of A.

(c) Is your answer in part (b) unique? Justify your conclusion.

7. (8 points) Determine whether each function below is a linear function $f : \mathbb{R}^2 \to \mathbb{R}^3$. If f is linear, show this by writing $f(x) = Ax$ for an appropriate matrix A. If f is not linear, find particular vectors and scalars for which f fails to be linear.

(a) $f(x_1, x_2) = (\frac{2x_1 - x_2}{2}, \frac{-x_1 + 2x_2}{2}, \frac{x_1 + x_2}{2})$

(b) $f(x_1, x_2) = (1 + x_1, 2 + x_2, 0)$