Fall 2024 Math F314X

Linear Algebra: Midterm 1

Name: Solutins

Rules:

- Show your work.
- You may have a single handwritten sheet of paper with writing on one side.
- You may use a calculatior

Problem	Possible	Score
1	10	
2	20	
3	15	
4	25	
5	30	
Total	100	

1. (10 points) Let a and b be n-vectors. Show by direct computation that $\|a+b\|^2 - \|a-b\|^2 = 4a^Tb$.

$$\|a+b\|^2 - \|a-b\|^2 = (a+b)^T(a+b) - (a-b)^T(a-b)$$
$$= a^Ta + 2a^Tb + b^Tb - [a^Ta - 2a^Tb + b^Tb]$$
$$= 4a^Tb$$

2. (20 points) Let $a_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $a_2 = (\frac{-\sqrt{3}}{2}, \frac{1}{2})$.

(a) Show that a_1 and a_2 are an **orthonormal** set of vectors.

- check $a_1 \perp a_2$

$$a_1^Ta_2 = \tfrac{1}{2}\left(-\tfrac{\sqrt{3}}{2}\right) + \tfrac{\sqrt{3}}{2}\left(\tfrac{1}{2}\right) = 0 \checkmark$$

- check normality

$$\|a_1\| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} = \sqrt{\tfrac{1}{4} + \tfrac{3}{4}} = 1 \qquad \checkmark$$

$$\|a_2\| = \sqrt{\left(-\tfrac{\sqrt{3}}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{3}{4} + \tfrac{1}{4}} = 1$$

(b) Write the vector $v = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$ as a linear combination of a_1 and a_2. Show your work.

- Find coefficients

$$v^Ta_1 = \begin{bmatrix} 4 \\ -8 \end{bmatrix}\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} = 2 - 4\sqrt{3}$$

$$v^Ta_2 = \begin{bmatrix} 4 \\ -8 \end{bmatrix}\begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix} = -2\sqrt{3} - 4$$

- answer

$$\begin{bmatrix} 4 \\ -8 \end{bmatrix} = (2 - 4\sqrt{3})\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} + (-2\sqrt{3} - 4)\begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

3. (15 points) Given the vectors $x_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(a) Carefully explain why no computation is necessary to determine that the vectors x_1, x_2, x_3, and x_4 are linearly dependent.

Since the number of vectors, 4, is larger than the dimension of the vectors, 3, our Fact A implies the must be linearly dependent.

(b) Use the definition to show the vectors x_1, x_2, x_3, x_4 are linearly dependent.

I need to find $\beta_1, \beta_2, \beta_3, \beta_4$ where $\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 = 0$.

Or,

$$\beta_1 + 2\beta_2 + \beta_3 = 0 \quad (i)$$
$$3\beta_1 + 2\beta_2 = 0 \quad (ii)$$
$$\beta_4 = 0 \quad (iii)$$

Demonstration of correctness

$$-2\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

Pick $\beta_2 = 3$. Then (ii) implies $\beta_1 = -2$. Thus, (i) implies

$$\beta_3 = -\beta_1 - 2\beta_2$$
$$= -(-2) - 2(3) = 2 - 6$$
$$= -4$$

4. (25 points) Let $T = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \right\}$.

(a) Show that T is linearly independent.

Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$. $(*)$

We need to show $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Equation $*$ gives the system of equations

$$\begin{aligned} \alpha_1 - \alpha_2 \quad\quad &= 0 \quad (i) \\ 2\alpha_1 \quad + 3\alpha_3 &= 0 \quad (ii) \\ 2\alpha_1 \quad + 4\alpha_3 &= 0 \quad (iii) \end{aligned}$$

Using (iii) - (ii) we get $\alpha_3 = 0$.

Plugging $\alpha_3 = 0$ into (ii) implies $\alpha_1 = 0$.

Plugging $\alpha_1 = 0$ into (i) implies $\alpha_2 = 0$.

Thus we have demonstrated that the only solution to $(*)$ is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus, we have shown using the definition that v_1, v_2, and v_3 are linearly independent.

(b) Begin the Gram-Schmidt algorithm on the vectors v_1, v_2, v_3, in the given order.

Recall $T = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \right\}$.

i. Find $\overline{q_1}$ and q_1.

$$\overline{q}_1 = v_1 = (1,2,2)$$

$$q_1 = \frac{\overline{q}_1}{\|q_1\|} = \frac{1}{\sqrt{1^2+2^2+2^2}} \cdot v_1 = \frac{1}{3} v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

I do a quick check that $\overline{q}_2 \perp v_1$ ✓

ii. Find $\overline{q_2}$ and q_2.

$$\overline{q}_2 = v_2 - \frac{v_2^T \overline{q}_1}{\|\overline{q}_1\|^2} \cdot \overline{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} - \frac{(-1)}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/9 \\ 2/9 \\ 2/9 \end{bmatrix}$$

$$= \begin{bmatrix} -8/9 \\ 2/9 \\ 2/9 \end{bmatrix}$$

$$\|\overline{q}_2\| = \sqrt{\frac{64+4+4}{81}} = \sqrt{\frac{72}{81}} = \frac{6\sqrt{2}}{9} = \frac{2\sqrt{2}}{3}$$

$$q_2 = \frac{3}{2\sqrt{2}} \begin{bmatrix} -8/9 \\ 2/9 \\ 2/9 \end{bmatrix} = \begin{bmatrix} -4/3\sqrt{2} \\ 1/(3\sqrt{2}) \\ 1/3\sqrt{2} \end{bmatrix}$$

I do a quick check that this has length 1

$$\frac{16+1+1}{18}$$ ✓

5. (30 points) Short Answer

(a) Let $a_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 5 \\ 4 \\ 3 \\ c \end{bmatrix}$ where c is some real number. Determine the values of c such that the angle between a_1 and a_2 be obtuse. (An angle is obtuse if it is larger than 90 degrees.)

We need $a_1^T a_2 < 0$.

$a_1^T a_2 = 5 + 8 - 9 + 4c = 4 + 4c$.

If $4+4c<0$, then $c<-1$.

Answer c can be any real number in $(-\infty, -1)$.

(b) If $\|x\| = 10$ and $\|y\| = 5$, what are possible values for $\|x + y\|$?

triangle inequality implies $\|x+y\| \leq \|x\| + \|y\| = 10+5 = 15$.

Since $\|x+y\| \geq 0$, we know $\|x+y\|$ is in $[0, 15]$.

In fact we can conclude that $\|x+y\| \geq \|x\| - \|y\| = 5$

using the observation that $\|x\| = \|x+y-y\| \leq \|x+y\| + \|y\|$.

(c) The vectors $v_1, v_2, \cdots, v_k$ are k- vectors and they form an orthonormal set of vectors. Can you conclude that they are a **basis**? **Justify** your conclusion using complete sentences.

Yes. They do form a basis.

Justification

① Because they are orthonormal, we know they are linearly independent.

② From ①, we know they are a set of k linearly independent k-vectors, which is the definition of a basis

(d) Suppose $f(x)$ is a scalar-valued function of a 50-vector that outputs the sum the first 30 values in the input vector x. Determine whether $f(x)$ is a linear function and demonstrate your answer is correct.

f is linear.

It is sufficient to demonstrate that

$$f(x) = a^T x$$ for some appropriate a.

In this case, I choose $\boxed{a = (1_{30}, 0_{20})}$ will work.

(e) Determine a matrix A such that $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_5 \\ x_4 \\ x_3 \end{bmatrix}$

A must be a 3×5 matrix. I reverse-engineer it.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_5 \\ x_4 \\ x_3 \end{bmatrix}$$ ✓