Fall 2024 *Math F314X*

Linear Algebra: Midterm 1

Name: ____________________

Rules:

- Show your work.
- You may have a single handwritten sheet of paper with writing on one side.
- You may use a calculatior

Problem	Possible	Score
1	10	
2	20	
3	15	
4	25	
5	30	
Total	100	

1. (10 points) Let a and b be n-vectors. Show by direct computation that $\|a+b\|^2 - \|a-b\|^2 = 4a^T b$.

2. (20 points) Let $a_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $a_2 = (\frac{-\sqrt{3}}{2}, \frac{1}{2})$.

   (a) Show that a_1 and a_2 are an **orthonormal** set of vectors.

   (b) Write the vector $v = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$ as a linear combination of a_1 and a_2. Show your work.

3. (15 points) Given the vectors $x_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(a) Carefully explain why no computation is necessary to determine that the vectors $x_1, x_2, x_3,$ and x_4 are linearly dependent.

(b) Use the definition to show the vectors x_1, x_2, x_3, x_4 are linearly dependent.

4. (25 points) Let $T = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \right\}$.

(a) Show that T is linearly independent.

(b) Begin the Gram-Schmidt algorithm on the vectors $v_1, v_2, v_3,$ in the given order.

Recall $T = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \right\}.$

i. Find $\overline{q_1}$ and q_1.

ii. Find $\overline{q_2}$ and q_2.

5. (30 points) Short Answer

(a) Let $a_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 5 \\ 4 \\ 3 \\ c \end{bmatrix}$ where c is some real number. Determine the values of c such that the angle between a_1 and a_2 be obtuse. (An angle is obtuse if it is larger than 90 degrees.)

(b) If $\|x\| = 10$ and $\|y\| = 5$, what are possible values for $\|x + y\|$?

(c) The vectors $v_1, v_2, \cdots, v_k$ are k- vectors and they form an orthonormal set of vectors. Can you conclude that they are a **basis**? **Justify** your conclusion using complete sentences.

(d) Suppose $f(x)$ is a scalar-valued function of a 50-vector that outputs the sum the first 30 values in the input vector x. Determine whether $f(x)$ is a linear function and demonstrate your answer is correct.

(e) Determine a matrix A such that $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_5 \\ x_4 \\ x_3 \end{bmatrix}$