Fall 2024 Math F314X

Linear Algebra: Midterm 1

Name: Solutions

Rules:

- Show your work.
- You may have a single handwritten sheet of paper with writing on one side.
- You may use a calculator

Problem	Possible	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20 points) Let $a = (\overset{a_1}{1}, \overset{a_2}{-1})$ and $x = (x_1, x_2, \cdots, x_{20})$. Let $b = a * x$, the convolution of the 2-vector a and the 20-vector x.

   (a) What are the dimensions of b?

   $$20 + 2 - 1 = 21$$

   (b) Find the entries of b. (Note that b is too long to list all of the entries. It might be useful to let $b = (b_1, b_2, \cdots, b_n)$ and describe b_i for appropriate i-values.)

   $b_1 = x_1$

   $b_2 = x_2 - x_1$

   $b_3 = x_3 - x_2$

   $\vdots$

   $b_{20} = x_{20} - x_{19}$

   $b_{21} = -x_{20}$

   b/c $\quad b_k = a_1 x_k + a_2 x_{k-1}$

   $\quad = 1 \cdot x_k + (-1) x_{k-1}$

   $\quad = x_k - x_{k-1}$

   (c) We know there is a matrix A such that $Ax = a * x$.

   i. What are the dimensions of the matrix A.

   21×20

   $$\underset{(n+1)\times n}{A}\ \underset{n\times 1}{x} = \underset{(n+1)\times 1}{a * x}$$

   ii. Describe the first three columns of A.

$$\begin{array}{c c} & \\ \text{21 rows} & \begin{bmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & \\ 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots \end{bmatrix} \\ & \begin{array}{cccc} \text{col 1} & \text{col 2} & \text{col 3} & \end{array} \end{array}$$

$P'(x) = C_2 + 2C_3 x$

2. (20 points) The 3-vector c represents the coefficients of the quadratic polynomial $p(x) = c_1 + c_2 x + c_3 x^2$.

   (a) Find a matrix B such that $Bc = \int_0^2 p(x)\, dx$.

$$\int_0^2 (c_1 + c_2 x + c_3 x^2)\, dx = c_1 x + \tfrac{1}{2} c_2 x^2 + \tfrac{1}{3} c_3 x^3 \Big]_0^2 = 2c_1 + 2c_2 + \tfrac{8}{3} c_3 = \cdots$$

$$\cdots = \begin{bmatrix} 2 & 2 & \tfrac{8}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \quad \text{So } B = \begin{bmatrix} 2 & 2 & \tfrac{8}{3} \end{bmatrix}$$

   (b) Express the conditions $p(0) = 0$, $p'(0) = 1$, $p(2) = -3$, and $p'(2) = 0$ as a set of linear equations of the form $Ac = b$.

$$\begin{aligned} p(0) &= c_1 &&= 0 \\ p'(0) &= c_2 &&= 1 \\ p(2) &= c_1 + 2c_2 + 4c_3 &&= -3 \\ p'(2) &= c_2 + 4c_3 &&= 0 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}}_{b}$$

   (c) Does A have a left inverse? Justify your answer. (Note: You are not asked to find a left inverse, only determine whether or not one exists.)

   Yes. The columns of A are linearly independent.

   (d) Does A have a right inverse? Justify your answer.

   No. As a tall matrix, its rows must be linearly dependent. Thus, it has no right inverse.

3. (20 points) Suppose $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$.

(a) Suppose someone, say Jill, gives you a matrix Q and a matrix R and asserts they are a QR-factorization of A. What do you need to check to confirm Jill is correct? Be specific and be efficient.

① Check $QR = A$

② Check R is upper triangle.

③ Check Q is orthogonal. (So check $Q^TQ = I$)

(b) In fact, A does have QR-factorization where $Q = \begin{bmatrix} 1 & 0 & 0 \\ \star & \sqrt{2}/2 & \sqrt{2}/2 \\ \star & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 1 & \star \\ 0 & \sqrt{2} & \star \\ 0 & \star & \star \end{bmatrix}$.

Fill in the blank spaces in the entries of Q and R below. You can show your work and/or explain your reasoning in the remaining space.

Answer:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{0} & \sqrt{2}/2 & \sqrt{2}/2 \\ \boxed{0} & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 & \boxed{0} \\ 0 & \sqrt{2} & \boxed{\sqrt{2}/2} \\ 0 & \boxed{0} & \boxed{\frac{3\sqrt{2}}{2}} \end{bmatrix}.$$

$\leftarrow a_3^T q_1$

$\leftarrow a_3^T q_2$

$\leftarrow a_3^T q_3$

↑ from Gram-Schmidt

↑ b/c R is upper Δ

$a_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ $\quad a_3^T q_1 = 0 \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \sqrt{2}/2 = a_3^T q_2$

$$a_3^T q_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = 3\sqrt{2}/2$$

4. (20 points) Let $A = \begin{bmatrix} 2 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find a left inverse of A. (You can do this by inspection.)

$$\begin{bmatrix} \frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- col 1 must be $\begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$
- row 1 must be $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$
- the rest are easy

(b) Does A have a right inverse? Justify your answer. (You are not asked to find one, just to determine whether or not one exists.)

Yes. A is square. So left invertibility implies right invertibility.

or check $\longrightarrow$ $\begin{bmatrix} 2 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

(c) Let $b = (4. -6, 3)$.

i. Use your answer in part (a) to solve the system of equations $Ax = b$.

$Ax = b \iff x = A^{-1}b.$

$$x = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-3 \\ 3+\frac{3}{2} \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{9}{2} \\ 3 \end{bmatrix}$$

ii. Since A happens to be upper triangular, describe an alternative method for solving $Ax = b$. (You don't have to perform the method, though you could use it to check your answer.)

Back substitution.

5. (20 points)

all entities are distinct

(a) Give an example of a 3 by 3 matrix A such that all entries are non-zero and such that A is **NOT** invertible. Explain why your example is not invertible.

$$A = \begin{bmatrix} 1 & 10 & \pi \\ 2 & 20 & \sqrt{2} \\ 3 & 30 & e \end{bmatrix}$$ is not invertible b/c $10 \cdot col_1(A) = col_2(A)$.

(b) Assume A is square, has linearly independent columns, and has a QR factorization. Does A^{-1} always exist? If so, find A^{-1} in terms of Q and R. If not, explain why A^{-1} may not exist.

$A = QR$; So $A^{-1} = (QR)^{-1} = R^{-1}Q^{-1}$

$= R^{-1}Q^{T}$ (since Q is orthogonal.)

(c) Suppose B is an orthogonal matrix.

i. State the **definition** of an orthogonal matrix.

It is square. It's columns form an orthonormal set of vectors. So they are orthogonal and length 1.

ii. Are the rows of A linearly independent? Justify your answer.

Yes $B^T B = I_n$. So B^T is Right invertible.

So its rows are lin. indep.

(d) Suppose A is an $m \times n$ matrix. Suppose D is an $m \times m$ diagonal matrix such that $D_{ii} = 2^i$. Describe the result of matrix multiplication DA.

DA is an $m \times n$ matrix such that the i^{th} row of DA is $2^i \cdot (row_i A)$.