Fall 2024 Math F314X

Linear Algebra: Midterm 2

Name: ____________________

Rules:

- Show your work.
- You may have a single handwritten sheet of paper with writing on one side.
- You may use a calculator

Problem	Possible	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20 points) Let $a = (1, -1)$ and $x = (x_1, x_2, \cdots, x_{20})$. Let $b = a * x$, the convolution of the 2-vector a and the 20-vector x.

   (a) What are the dimensions of b?

   (b) Find the entries of b. (Note that b is too long to list all of the entries. It might be useful to let $b = (b_1, b_2, \cdots, b_n)$ and describe b_i for appropriate i-values.)

   (c) We know there is a matrix A such that $Ax = a * x$.

      i. What are the dimensions of the matrix A.

      ii. Describe the first three columns of A.

2. (20 points) The 3-vector c represents the coefficients of the quadratic polynomial $p(x) = c_1 + c_2 x + c_3 x^2$.

   (a) Find a matrix B such that $Bc = \int_0^2 p(x)\,dx$.

   (b) Express the conditions $p(0) = 0$, $p'(0) = 1$, $p(2) = -3$, and $p'(2) = 0$ as a set of linear equations of the form $Ac = b$.

   (c) Does A have a left inverse? Justify your answer. (Note: You are not asked to find a left inverse, only determine whether or not one exists.)

   (d) Does A have a right inverse? Justify your answer.

3. (20 points) Suppose $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$.

(a) Suppose someone, say Jill, gives you a matrix Q and a matrix R and asserts they are a QR-factorization of A. What do you need to check to confirm Jill is correct? Be specific and be efficient.

(b) In fact, A does have QR-factorization where $Q = \begin{bmatrix} 1 & 0 & 0 \\ \star & \sqrt{2}/2 & \sqrt{2}/2 \\ \star & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 1 & \star \\ 0 & \sqrt{2} & \star \\ 0 & \star & \star \end{bmatrix}$.

Fill in the blank spaces in the entries of Q and R below. You can show your work and/or explain your reasoning in the remaining space.

Answer:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{} & \sqrt{2}/2 & \sqrt{2}/2 \\ \boxed{} & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 & \boxed{} \\ 0 & \sqrt{2} & \boxed{} \\ 0 & \boxed{} & \boxed{} \end{bmatrix}.$$

4. (20 points) Let $A = \begin{bmatrix} 2 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find a left inverse of A. (You can do this by inspection.)

(b) Does A have a right inverse? Justify your answer. (You are not asked to find one, just to determine whether or not one exists.)

(c) Let $b = (4, -6, 3)$.

i. Use your answer in part (a) to solve the system of equations $Ax = b$.

ii. Since A happens to be upper triangular, describe an alternative method for solving $Ax = b$. (You don't have to perform the method, though you could use it to check your answer.)

5. (20 points)

   (a) Give an example of a 3 by 3 matrix A such that all entries are non-zero, all entries of A are distinct (no repeats) and such that A is **NOT** invertible. Explain why your example is not invertible.

   (b) Assume A is square, has linearly independent columns, and has a QR factorization. Write its inverse A^{-1} in terms of Q^T and R^{-1}.

   (c) Suppose B is an orthogonal matrix.

      i. State the **definition** of an orthogonal matrix.

      ii. Are the rows of B linearly independent? Justify your answer.

   (d) Suppose A is an $m \times n$ matrix. Suppose D is an $m \times m$ diagonal matrix such that $D_{ii} = 2^i$. Describe the result of matrix multiplication DA.