

Page 203 Invertibility Conditions

A is $n \times n$ matrix.

The following are equivalent

- A is left-invertible (or A has a left inverse)
- A is right-invertible (or A has a right inverse)
- A is invertible
- The columns of A are linearly independent
- The rows of A are linearly independent

Crucial piece: A has a left inverse \Rightarrow
 A has a right inverse

Logic: A has left inverse \Rightarrow

cols of A are linearly independent \Rightarrow
cols of A are a basis (b/c n n -vectors) \Rightarrow

for every $i=1,2,\dots,n$, e_i is a linear
combination of cols A . \iff

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \beta_1 \text{col}_1 A + \beta_2 \text{col}_2 A + \dots + \beta_n \text{col}_n A$$
$$= \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_n a_n \quad a_i = \text{col}_i A \quad \iff$$

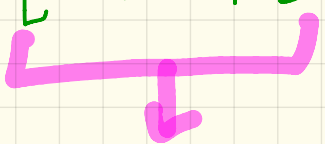
and similar for e_2, e_3, \dots, e_n

$$\Leftrightarrow \left[\begin{array}{c|c|c|c} | & | & | & | \\ a_1 & a_2 & \dots & a_n \\ | & | & | & | \end{array} \right] \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = A \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = Ab_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_1$$

$$Ab_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, Ab_3 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \dots, Ab_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{c|c|c|c} | & | & | & | \\ Ab_1 & Ab_2 & \dots & Ab_n \\ | & | & | & | \end{array} \right] = I_n$$

$$\Leftrightarrow A \cdot \left[\begin{array}{c|c|c|c} | & | & | & | \\ b_1 & b_2 & \dots & b_n \\ | & | & | & | \end{array} \right] = I_n$$



A^{-1} or a left-inverse
for A .

Return to #6C from worksheet

The inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ is $C = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$
 $= \begin{bmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{bmatrix}$

Find b_1, b_2, b_3 .

$$A \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{array}{l} \beta_1 + \beta_2 + 2\beta_3 = 1 \quad \text{argmt} \\ \beta_1 + \beta_2 + \beta_3 = 0 \quad \Rightarrow \\ 2\beta_1 + 3\beta_2 + 4\beta_3 = 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right] \Rightarrow b_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

How does this look different to find b_2 ?

Pure math algorithm/argument:

$$[A \mid I_3] \xrightarrow{\text{rref}} [I_3 \mid A^{-1}]$$