Page 203 Invertibility Conditions

A is $n \times n$ matrix.

The following are equivalent

- A is left-invertible (or A has a left invers)
- A is right-invertible (or A has a right inverse)
- A is invertible
- The columns of A are linearly independent
- The rows of A are linearly independent

<u>Crucial piece</u>: A has a left inverse $\Rightarrow$ A has a right inverse

<u>Logic</u>: A has left inverse $\Rightarrow$
cols of A are linearly indepdent $\Rightarrow$
cols of A are a <u>basis</u> (b/c n n-vectors) $\Rightarrow$

for every $i = 1, 2, .., n$, e_i is a linear combination of cols A. $\Leftrightarrow$

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \beta_1 \, col_1 A + \beta_2 \cdot col_2 A + .. \beta_n \, col_n A$$

$$= \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_n a_n \qquad a_i = col_i A$$

and similar for $e_2, e_3, \dots, e_n$ $\Leftrightarrow$

$$\Longleftrightarrow \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = A \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = Ab_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_1$$

$$Ab_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad Ab_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \ldots, \ Ab_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_n \\ | & | & & | \end{bmatrix} = I_n$$

$$\Longleftrightarrow A \cdot \underbrace{\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}}_{} = I_n$$

A^{-1} or a left inverse for A.

Return to #6c from worksheet

The inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ is $C = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix}$

Find b_1, b_2, b_3.

$A \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \iff \begin{aligned} \beta_1 + \beta_2 + 2\beta_3 &= 1 \\ \beta_1 + \beta_2 + \beta_3 &= 0 \\ 2\beta_1 + 3\beta_2 + 4\beta_3 &= 0 \end{aligned} \xrightarrow{\text{augment}}$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow b_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

How does this look different to find b_2?

Pure math algorithm/argument:

$[A \,\vdots\, I_3] \xrightarrow{\text{rref}} [I_3 \,\vdots\, A^{-1}]$