Fri Nov 8

- Midterm 2 Monday
  The review sheet is helpful.

From Wednesday

- Given system $Ax=b$, columns of A are linearly independent

  the least squares approximation is, $\hat{x}$,

  where $\hat{x} = A^{+} b$ and think of $\hat{x}$ as

  - the vector that minimizes $\|Ax-b\|^2$

    or

  - a "best" approximation of a solution to $Ax=b$

- <u>Terminology</u> $f(x) = \|Ax-b\|^2$ ← objective fcn.

  - $(A^TA)^{-1}A^Tb = \hat{x} \Leftarrow A^{+}b = \hat{x}$

  - To solve $Ax = b$. Mult. by A^T on both.

normal equations → $\boxed{A^TAx = A^{+}b}$

Example: Find the line of best fit for the points

$(1,1), (2,4), (4,11)$.

We want $y = mx + b$ that is close to points. So ...

we want m and b so that

$$1 = m + b$$
$$4 = 2m + b$$
$$11 = 4m + b.$$

or $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 11 \end{bmatrix}$. There is No solution b/c $\frac{\Delta y}{\Delta x}$ is not constant for our points.

Find a least squares approx. solution.

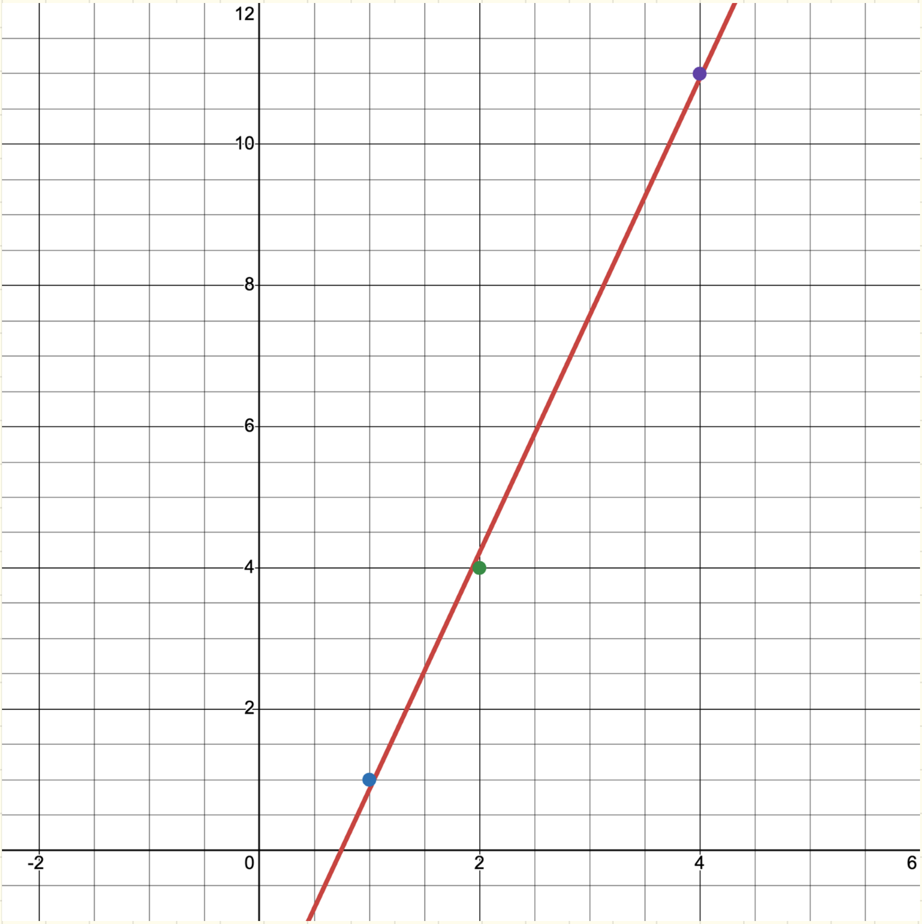
If $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix}$, then $A^{+} = \left(\begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \left(\begin{bmatrix} 21 & 7 \\ 7 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{63-49} \begin{bmatrix} 3 & -7 \\ -7 & 21 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{14} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2/7 & -1/14 & 5/14 \\ 1 & 1/2 & -1/2 \end{bmatrix}.$$

$$A^{+} \begin{bmatrix} 1 \\ 4 \\ 11 \end{bmatrix} = \begin{bmatrix} 47/14 \\ -5/2 \end{bmatrix}$$

$\boxed{\text{So } y = \frac{47}{14}x - \frac{5}{2}}$ $\longrightarrow$



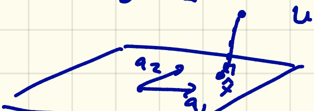
Desmos

How confident that this would work for 100 points?

Aside: $u = A\hat{x} - b = \begin{bmatrix} -1/7 \\ 3/14 \\ -1/14 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix}$

$u^T a_1 = \frac{-1}{7} + \frac{6}{14} - \frac{4}{14} = 0$

$u^T a_2 = \frac{-2}{14} + \frac{3}{14} - \frac{1}{14} = 0$



There is another strategy for finding A^+:

using QR factorization.

<u>Quick Review</u> Given A an $n \times n$ matrix w/ linearly independent columns,

(i) use Gram-Schmidt to find orthonormal vectors $q_1 q_2 \cdots q_n$

which form the columns of matrix Q, an orthogonal matrix.

(ii) Find R using $R_{ij} = a_j^T q_i$ for $1 \le i \le j \le n$ and zeros below diagonal

What if A is not square? Say $m \times n$.

Given A an $m \times n$ matrix w/ linearly independent columns, ← well, then $m > n$

(i) use Gram-Schmidt to find orthonormal vectors $q_1 q_2 \cdots q_n$ || We can still apply GS to get the q_i's

which form the columns of matrix Q, an ~~orthogonal matrix~~. ← Q would also be $m \times n$ and have orthonormal columns.

(ii) Find R using $R_{ij} = a_j^T q_i$ for $1 \le i \le j \le n$ and zeros below diagonal ← This still works b/c we have n q_i's and n a_i's!

Ex | QR-decomposition/factorization for

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix}. \quad \xrightarrow{\text{work}} \ldots$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{21}} & \sqrt{6}/3 \\ \frac{2}{\sqrt{21}} & \sqrt{6}/6 \\ \frac{4}{\sqrt{21}} & -\frac{\sqrt{6}}{6} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{21} & \sqrt{21}/3 \\ 0 & \sqrt{6}/3 \end{bmatrix}$$

(Check it's correct!)

<u>You</u> Use $(XY)^T = Y^T X^T$ and $(XY)^{-1} = Y^{-1} X^{-1}$

to show

If $A = QR$, then $A^+ = R^{-1} Q^T$.

<u>Answer</u>:

$$\begin{aligned} A^+ &= (A^T A)^{-1} A^T \\ &= ((QR)^T QR)^{-1} (QR)^T \\ &= (R^T Q^T Q R)^{-1} (R^T Q^T) \\ &= (R^T \cdot R)^{-1} (R^T Q^T) \\ &= R^{-1} (R^T)^{-1} R^T Q^T \\ &= R^{-1} Q^T \end{aligned}$$

$Q^T Q = I_n$

$(R^T)^{-1} R = I_n$

As a matter of implementation:

Goal: Find $\hat{x}$ the least squares approx of

$Ax=b$. (Columns of A are lin. indep)

QR method.

① $A = QR$. ← find QR factorization

② Solve $Rx = Q^T b$ by back sub.

To be specific: Find $\hat{x}$ for $\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 11 \end{bmatrix}}_{b}$

$\frac{\sqrt{6}}{3} + \frac{4\sqrt{6}}{6} - \frac{11\sqrt{6}}{6}$

$\frac{2\sqrt{6}}{6} - 7$

① $A = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{21}} & \sqrt{6}/3 \\ \frac{2}{\sqrt{21}} & \sqrt{6}/6 \\ \frac{4}{\sqrt{21}} & -\frac{\sqrt{6}}{6} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{21} & \sqrt{21}/3 \\ 0 & \sqrt{6}/3 \end{bmatrix}}_{R}$. So Solve $Rx = Q^T b$ or

$Q^T b = \begin{bmatrix} 53/\sqrt{21} \\ \frac{-5\sqrt{6}}{6} \end{bmatrix}$

$$\sqrt{21}\, x_1 + \frac{\sqrt{21}}{3} x_2 = 53/\sqrt{21}$$

$$\frac{\sqrt{6}}{3} x_2 = \frac{-5\sqrt{6}}{6} \implies x_2 = \frac{-5}{2}$$

Plug $x_2 = \frac{-5}{2}$ into $\sqrt{21}\, x_1 + \frac{\sqrt{21}}{3} x_2 = \frac{53}{\sqrt{21}}$ to get $x_1 = \frac{47}{14}$

How to find the point on the line L closest to the point $P(1,3,5)$ if L is the line through $(0,0,0)$ in the direction

$$\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Want x so that

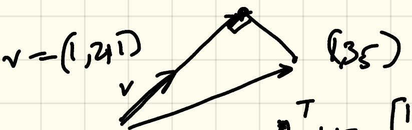
$$\begin{bmatrix}1\\2\\1\end{bmatrix} \underset{\substack{\uparrow\\ \text{number}}}{x} = \begin{bmatrix}1\\3\\5\end{bmatrix}$$

no soln. Find $\hat{x}$.

$$\hat{x} = A^{+}\begin{bmatrix}1\\3\\5\end{bmatrix} = \left(\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}1\\3\\5\end{bmatrix}$$

$$= (6)^{-1}\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}1\\3\\5\end{bmatrix} = \frac{1}{6}(1+6+5) = 2$$

$\hat{x} = 2$ or $\begin{bmatrix}2\\4\\2\end{bmatrix}$ is closest to $\begin{bmatrix}1\\3\\5\end{bmatrix}$



$$\begin{bmatrix}2\\4\\2\end{bmatrix} - \begin{bmatrix}1\\3\\5\end{bmatrix} = \begin{bmatrix}1\\1\\-3\end{bmatrix} = w$$

$$A^T w = \begin{bmatrix}1\\2\\1\end{bmatrix}^T\begin{bmatrix}1\\1\\-3\end{bmatrix} = 0 \checkmark$$