From Monday

def convolution of n-vector a and m-vector b produces $(n+m-1)$-vector c where

$$c_k = \sum_{i+j=k+1} a_i b_j \quad , \text{ denoted } c = a * b$$

We see this is analogous

- think of $a*b$ through lens of polynomial multiplication

- Ex $a = (1,2)$, $b = (-1,3)$

  Find c by thinking of a, b, c as coeff. vectors of polys.

  $a \to 1 + 2x.$ Find c by

  $b \to -1 + 3x$

  $$(1+2x)(-1+3x)$$
  $$= -1 + x + 6x^2$$

  So $c = (-1, 1, 6)$

- $a = (a_1, a_2, \ldots, a_n) \rightsquigarrow p(x) = a_1 + a_2 x + x_3 x^2 + x_4 x^3 + \ldots + a_n x^{n-1}$

  $b = (b_1, \ldots, b_m) \rightsquigarrow q(x) = b_1 + b_2 x + \ldots + b_m x^{m-1}$

  If c is coeff vector of $p(x) q(x)$ then

  - c has entries from deg 0 to deg $m+n-2$ so $m+n-1$ entries
  - c_k = coeff of x^{k+1} we need to sum all $a_i b_j$ where $i+j = k+1$.

Illustrates certain obvious propertie

$$a * b = b * a$$

$$a * (b * c) = (a * b) * c$$

$a * b = 0$ if and only if $a = 0$ or $b = 0$

$a*b$ can be represented by matrix·vector product. (!!)

$a=(1,2,3)$, $b=(-1,2)$

Let $T(a) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \\ 0 & 3 \end{bmatrix}$. Then $a*b = \underset{4\times 2}{T(a)} \cdot \underset{2\times 1}{b} = \underset{4\times 1}{c}$

Toeplitz

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} +1(-1)+0 \\ 2(-1)+1(2) \\ 3(-1)+2\cdot 2 \\ 0\cdot(-1)+3\cdot 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 6 \end{bmatrix}$$

You find $T(b)$ so that $a*b = T(b)a$

$$\begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 6 \end{bmatrix}$$

* So polynomial expansion can be modeled by matrix-vector product.

* Convolution appears when studying time series

$x = (x_1, x_2, \ldots, x_n)$ is a time series means x_i's are the same type of measurement over time (typically equal intervals).

- high temp in Fbks each day for 50 yrs
- height of human each year for 50 years
- conc. of drug in bloodstream each min for 24 hrs
- audio signal over 17 sec.

Ex| Say x = daily high temps in Fbks for 10 consec. years

So x is 3650-vector.

Say $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

What is $a * x$? → notation/math → wordy explanation math : c is a $3650+3-1=3652$-vector

(mostly)

If $3 \le k \le 3650$ $\quad c_k = \frac{1}{3}(x_{k-2}+x_{k-1}+x_k)$

(little exception)

$c_1 = \frac{1}{3}x_1$, $c_2 = \frac{x_1+x_2}{3}$, $c_{3651} = \frac{x_{3650}+x_{3649}}{3}$

$c_{3652} = \frac{1}{3}x_{3650}$

temp

Words : c is the awerge high for previous 3 days. So c is a "running average."

Smooths the data set.

Input-output systems or

system-impulse response

- high temps in AK range (input)
  high river levels in Tanana (output) @ Fbks

- change in human behavior due to covid.

- rainfall & river height

  x = rainfall in inches each day
  y = inches above normal river height in
  h = (0, 1, 2)

  Input x = (0, 4, 0, 0..)

  Find and intrpt y = h * x.

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ 2 & 1 & 0 & \vdots \\ 0 & 2 & 1 & \\ \vdots & 0 & 2 & \\ & \vdots & 0 & \\ & & \vdots & \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 8 \\ 0 \\ \vdots \end{bmatrix}$$

4in rains on day 2,

on day 3, river is 4in above

on day 4, river is 8 in above

on dy 3, back to normal