Ch 7 Applications of matrix-vector product.

Permutations

Ex A $n \times n$ matrix such that every row + every column has exactly one 1, All other entries are zero.
Called a permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ c \\ d \\ a \end{bmatrix}$$

special case of selector matrix

Geometry

Ex geometric trans. of plane

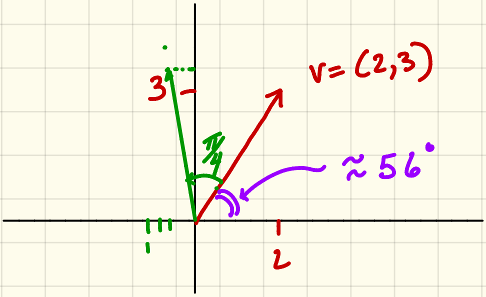
book says rotation by θ given by

$$\begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix}$$

means rotatn by $\frac{\pi}{4} = \theta$
$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

means rotate by $\frac{\pi}{4}=\theta$

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = A$$

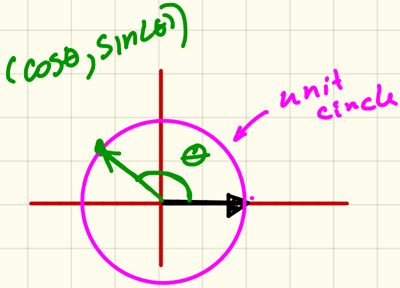


$$Av = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \frac{\sqrt{2}}{2} & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{2}/2 \\ 5\sqrt{2}/2 \end{bmatrix} \approx \begin{bmatrix} -0.7 \\ 3.5 \end{bmatrix}$$

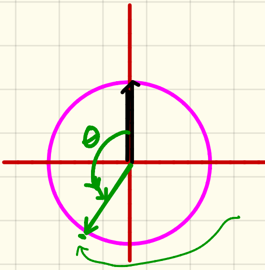
How? $\begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix}$

Strategy: If $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1$, then $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



$$\begin{bmatrix} \cos\theta & \\ \sin\theta & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

If $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2$, then $Ax = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$



$$\begin{bmatrix} \cos\theta & \\ \sin\theta & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\left(\cos(\theta + \pi/2), \sin(\theta + \tfrac{\pi}{2})\right) = \left(-\sin(\theta), \cos(\theta)\right)$$

Principle Determine column i of A by finding the image of e_i.

Ex Project $\mathbb{R}^2$ onto x-axis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Ex Reflect $\mathbb{R}^2$ about y-axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Down Sampling

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix}$$

Convolution

def: a is an n-vector
b is an m-vector
The convolution of a and b is an $(n+m-1)$-vector c denoted $\boxed{a*b}$

where $c_k = \sum_{i+j \le k+1} a_i b_j$.

Ex 1 $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3, b_4)$

$c = a*b$ is a $3+4-1 = 6$ vector

$k=1$ $c_1 = a_1 b_1$
$k=2$ $c_2 = a_1 b_2 + a_2 b_1$
$k=3$ $c_3 = a_1 b_3 + a_2 b_2 + a_3 b_1$
$k=4$ $c_4 = a_1 b_4 + a_2 b_3 + a_3 b_2 + \cancel{a_4 b_1}$
$k=5$ $c_5 = \cancel{a_1 b_5} + a_2 b_4 + a_3 b_3 + \cancel{a_4 b_2} + \cancel{a_5 b_1}$
$k=6$ $c_6 = a_3 b_4$

Ex $a = (\overset{a_1}{1}, \overset{a_2}{2}, \overset{a_3}{3})$ $b = (\overset{b_1}{-1}, \overset{b_2}{2})$

$$a*b = (\underset{\substack{k=1\\k+1=2}}{1 \cdot -1}, \underset{\substack{k=2\\k+1=3}}{1 \cdot 2 + 2(-1)}, \underset{\substack{k=3\\k+1=4}}{2 \cdot 2 + (3)(-1)}, \underset{\substack{k=4\\k+1=5}}{3 \cdot 2})$$

$$= (-1, 0, 1, 6) = c$$

$\boxed{\text{Ex}}$ $a=(\overset{a_1}{1}, \overset{a_2}{2}, \overset{a_3}{3})$ $b=(\overset{b_1}{-1}, \overset{b_2}{2})$, $(-1, 0, 1, 6) = c$

$$p(x) = 1+2x+3x^2, \quad q(x) = -1+2x$$

$$p(x)\,q(x) = (1+2x+3x^2)(-1+2x)$$

$$= (1)(-1) + (1\cdot 2 + 2(-1))x + ((-1)(3)+2\cdot 2)x^2 + 3\cdot 2x^3$$

$$= -1 + 0x + 1\cdot x^2 + 6x^3$$

$$a(x) = a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1}$$

$$b(x) = b_1 + b_2 x + b_3 x^2 + \ldots + b_m x^{m-1}$$

Then $a(x)\cdot b(x)$ is a polynomial of degree $n+m-1$ with coefficients $a*b$.

Illustrates certain obvious properties

$$a*b = b*a$$

$$a*(b*c) = (a*b)*c$$

$a*b=0$ if and only if $a=0$ or $b=0$.

$a*b$ can be represented by matrix·vector product. (!!)

$a=(1,2,3)$, $b=(-1,2)$

Let $\underset{\text{Toeplitz}}{T(a)} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \\ 0 & 3 \end{bmatrix}$. Then $a*b = \underset{4\times 2}{\underset{\nearrow}{T(a)}} \cdot \underset{2\times 1}{\underset{\uparrow}{b}} = \underset{4\times 1}{\underset{\uparrow}{c}}$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} +1(-1)+0 \\ 2(-1)+1(2) \\ 3(-1)+2\cdot 2 \\ 0\cdot(-1)+3\cdot 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 6 \end{bmatrix}$$

You find $T(b)$ so that $a*b = T(b)a$

$$\begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 6 \end{bmatrix}$$