Last of Ch8

Recall $f: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

① for all n-vectors x, y and constants α, β

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$ ← a.k.a superposition

or, equivalently

② $f(x) = Ax$ for some $m \times n$ matrix A

As in Ch2: Lots of things are called linear that are actually affine.

def: $f(x): \mathbb{R}^n \to \mathbb{R}^m$ is affine if

$f(x) = Ax + b$ for some $m \times n$ matrix A and some m-vector b

Ex $f: \mathbb{R}^4 \to \mathbb{R}^2$ defined as $f(x_1, x_2, x_3, x_4) = (2x_2, x_3 + 2)$

- f not linear: $x = (1,1,1,1)$, $y = 0$, $\alpha = 2$, $\beta = 0$

You

$$f(\alpha x + \beta y) = f(2,2,2,2) = (4,4)$$

$$\alpha f(x) + \beta f(y) = 2f(1,1,1,1) = 2(2,3) = (4,6)$$

if $f: \mathbb{R}^4 \to \mathbb{R}^2$ defined as $f(x_1, x_2, x_3, x_4) = (2x_2, x_3+2)$

Show f is affine by finding A and b.

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

- It is a fact that if $f(x)$ is affine, superposition holds in the special case where $\alpha + \beta = 0$.

<u>Pf</u>: Sppx $f(x) = Ax + b$, x, y arbitrary α, β constants s.t. $\alpha + \beta = 1$.

Then $f(\alpha x + \beta y) = A(\alpha x + \beta y) + b$

$$= \alpha Ax + \beta Ay + (\alpha + \beta) b$$

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$$= \alpha Ax + \alpha b + \beta Ay + \beta b$$
$$= \alpha (Ax + b) + \beta (Ay + b)$$
$$= \alpha f(x) + \beta f(y) \checkmark$$

Use the previous fact to show

$f(x_1, x_2) = (x_1 x_2, x_2)$ is <u>not</u> affine.

<u>Strategy:</u> Find counter-example.

$x = (1,1)$
$y = (0,0)$
$\alpha = 1/3$
$\beta = 2/3$

$\Big]\ \alpha + \beta = 1$

$$f(\alpha x + \beta y) = f(1/3, 1/3) = \left(\tfrac{1}{9}, \tfrac{1}{3}\right)$$

$$\alpha f(x) + \beta f(y) = \tfrac{1}{3} f(1,1) = \tfrac{1}{3}(1,1) = \left(\tfrac{1}{3}, \tfrac{1}{3}\right)$$