Ch 8 Linear & Affine Function (Again)

Ch 2: $f: \mathbb{R}^n \to \mathbb{R}$

$$f \text{ linear} \iff f(x) = a^T x \longleftrightarrow$$

Ex) $f(x)$ is the average of entries of x

$$f(x) = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$$

Choose $a = \left(\frac{1}{n}\right) \mathbf{1}_n = (1, 1, \ldots, 1)$

$f(x) = a^T x$ ✓ So linear.

Superposition
For all $\alpha, \beta \in \mathbb{R}$, $x, y \in \mathbb{R}^n$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

Ex) $f(x)$ outputs maximum entry in x.

Not linear

Choose $x = (1,0)$, $\alpha = -5$, $\beta = 0$, $y = (0,0)$

RHS $= f(-5,0) = 0$ not equal.

LHS $= -5(f(1,0)) = -5$

Ch8: linear functions (×2)

Ch6: $f: \mathbb{R}^n \to \mathbb{R}$

Ex: $f(x_1, x_2, x_3) = x_1 + 2x_2 - 3x_3$

$f(\underline{2,-1,1}) = 2 + 2(-1) - 3(1) = \underline{-3}$

$f: \mathbb{R}^n \to \mathbb{R}^m$ ← m-vectors

↑ n-vectors

Ex: $f(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3 - x_4)$

Find A

$f: \mathbb{R}^4 \to \mathbb{R}^2$

$f(1,2,3,4) = (1+2, 3-4) = (3,-1)$

Ch6

def: f is linear if for every constant α, β and every vector x, y

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$ ← super-position

$$\boxed{f: \mathbb{R}^n \to \mathbb{R}^m \text{ is linear}}$$

$$\boxed{f(\alpha x+\beta y) = \alpha f(x) + \beta f(y) \text{ for all vectors } x, y \text{, constants } \alpha, \beta}$$

clear

think about e_i's

$$\boxed{f(x) = Ax \text{ for } m \times n \text{ matrix } A}$$

These are all equivalent.

*Ch6 is just a special case where $m=1$.

Find A by:

$$A = \begin{bmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{bmatrix}$$

Ex

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ x_3-x_4 \end{bmatrix}$$

Examples that are not linear.

$f : \mathbb{R}^2 \to \mathbb{R}^2$

① $f(x_1, x_2) = (x_1^2, x_2)$

Show $\boxed{\text{superposition}}$ fails for

our choice

$x = (1, 0)$
$y = (0, 0)$
$\alpha = -5$
$\beta = 0$

$\alpha x + \beta y = -5(1,0) + 0(0,0)$

$= (-5, 0)$

$f(\alpha x + \beta y) = f(-5, 0) = ((-5)^2, 0) = \boxed{(25, 0)}$

$\alpha f(x) + \beta f(y) = -5 f(1,0) + 0 f(0,0)$

$= -5(1,0) = \boxed{(-5, 0)}$

not equal

② $f(x_1, x_2) = (x_1 + x_2, x_2 + 1)$

$x = (0,0)$ $\quad \alpha = 1$
$y = (0,0)$ $\quad \beta = 1$

$\alpha x + \beta y = (0,0)$
$f(\alpha x + \beta y) = (0,1)$
$1 f(0,0) + 1 f(0,0) = (0,1) + (0,1)$
$= (0,2)$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad x = (x_1, x_2) \quad y = (y_1, y_2)$$

$$\alpha = \text{scalar}$$

$$Ax = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \qquad Ay = \begin{bmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{bmatrix}$$

$$\boxed{Ax + Ay} = \begin{bmatrix} ax_1 + bx_2 + ay_1 + by_2 \\ cx_1 + dx_2 + cy_1 + dy_2 \end{bmatrix}$$

equal

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

$$\boxed{A(x+y)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} a(x_1+y_1) + b(x_2+y_2) \\ c(x_1+y_1) + d(x_2+y_2) \end{bmatrix}$$

$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}$ $\qquad \boxed{A(\alpha x)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}$

$$= \begin{bmatrix} a\alpha x_1 + b\alpha x_2 \\ c\alpha x_1 + d\alpha x_2 \end{bmatrix}$$

equal

$$\boxed{\alpha (Ax)} = \alpha\left(\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \right)$$

Observation

- $A(x+y) = Ax + Ay$
- $A(\alpha x) = \alpha Ax$
- $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$

looks like "superposition"

- Suppose $f: \mathbb{R}^2 \to \mathbb{R}^3$ is linear

and $f(e_1) = (1,0,1)$, $f(e_2) = (1,-1,0)$.

Determine $f(3, \sqrt{2})$.

Ans: $(3, \sqrt{2}) = 3e_1 + \sqrt{2}\, e_2$

So $f(3,\sqrt{2}) = f(3e_1 + \sqrt{2}\, e_2)$

$$= 3f(e_1) + \sqrt{2}\, f(e_2)$$

$$= 3(1,0,1) + \sqrt{2}\,(1,-1,0)$$

$$= (3,0,3) + (\sqrt{2}, -\sqrt{2}, 0)$$

$$= (3+\sqrt{2}, -\sqrt{2}, 3) \checkmark$$

- Find A so that $f(x) = Ax$.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$f: \mathbb{R}^n \to \mathbb{R}^m$ is <u>affine</u> if

$f(x) = Ax + b$ for $m \times n$ matrix A and m-vector b.

$\boxed{\text{Ex}}$ $f(x_1, x_2) = (x_1 + x_2, x_2 + 1)$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b}$$