Mon 18 Nov

Determinants

- We will appeal to the recursive/cofactor definition
- Use it.
- Learn some crucial properties.

Ground Rules: All matrices are now square.

Notation
- matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

- determinant of matrix A $= \det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

Motivating Examples

- If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = -2$

- If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, then $\det(A) = A_{11}A_{22} - A_{12}A_{21}$

- Where have we seen this before and what did it tell us? It was used to calculate A^{-1} $\boxed{\text{OR}}$ to indicate if no A^{-1} exists.

Recall: $A^{-1} = \dfrac{1}{A_{11}A_{22} - A_{12}A_{22}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$;

Motivating Properties (Observe these for 2x2 matrices)

0. If the rows are linearly dependent, then $\det(A)=0$.

1. $\det(I_2) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\cdot1-0\cdot0=1$

2. If two rows are exchanged ($r_i \leftrightarrow r_j$), then the sign (+/-) of the determinant changes.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad-bc$$

$$B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}, \quad \det(B) = cb-ad = -(ad-bc)$$

3. If k is a constant and A is 2×2 matrix, then $\det(kA) = k^2\det(A)$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix},$$

$$\det(kA) = k^2ad - k^2bc = k^2(ad-bc)$$

4. Adding/Subtracting a multiple of one row to another leaves the determinant unchanged

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{r_1 := r_1 + kr_2} \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix} = B$$

$$\det(B) = (a+kc)d - (b+kd)c = ad + kcd - bc - kcd$$

$$= ad - bc = \det(A) \checkmark$$

5. If A is upper/lower triangular, then $\det(A)$ is the product of the diagonal entries.

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix}$$

6. If $C = AB$, then $\det(C) = \det(A)\det(B)$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \quad C = AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$\det(C) = (ae+bg)(cf+dh) - (ce+dg)(af+bh)$$

$$= ad(eh-fg) + bc(fg-eh) + (aecf - aecf$$

$$+ bdgh - bdgh)$$

$$= (ad-bc)(eh-fg) = \det(A)\det(B)$$

Notable Consequences

7. If $\det(A) \neq 0$, then A^{-1} exists and $\det(A^{-1}) = \frac{1}{\det(A)}$.

uses 0., 1., 6.

Since $A^{-1}A = I_2$ and $\det(I_2) = 1$, then

$$1 = \det(I_2) = \det(A^{-1}A) = \det(A^{-1})\det(A).$$

Solve for $\det(A^{-1})$ to get $\det(A^{-1}) = \frac{1}{\det(A)}$.

8. Properties 2, 4, 5 suggest a strategy for computing $\det(A)$:

Use row exchanges and adding/subtracting one row from another to form a triangular matrix.

Baby Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \xrightarrow{r_2 := r_2 - r_1} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = B, \quad \det(A) \overset{4}{=} \det(B) \underset{5}{=} 1 \cdot 3 = 3$$

9 If $A = [a]$ is a 1×1 matrix, then

$$\det(A) = a.$$