The $n \times n$ case

def: Let A be an $n \times n$ matrix and A_{ij} is the number in i^{th} row and j^{th} column of A.

our book's notation

Let S_{ij} denote the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i^{th} row and the j^{th} column from A.
Typically, S_{ij} is called a <u>minor</u> of A, or, the ij'th minor of A.

Let $C_{ij} = (-1)^{i+j} \det(S_{ij})$ be the <u>cofactor</u> of A associated with element A_{ij}.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}, \quad A_{11} = 1, \quad S_{11} = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -2 & 3 \\ 0 & 4 \end{vmatrix} = -8$$

$\boxed{\text{You}}$ find C_{12} and C_{13}.

$$C_{12}: \; A_{12} = 2, \; S_{12} = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}, \; C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 3 \\ 2 & 4 \end{vmatrix} = (-1)(-4-6)$$
$$= 10$$

$$C_{13}: \; A_{13} = 3, \; S_{13} = \begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix}, \; C_{13} = (-1)^{1+3} \begin{vmatrix} -1 & -2 \\ 2 & 0 \end{vmatrix} = 4$$

Thm: A is $n \times n$ matrix with entries A_{ij}.

For any row i (so $i = 1, 2, \ldots, n$)

$$\det(A) = \sum_{j=1}^{n} A_{ij} C_{ij} = A_{i1} C_{i1} + A_{i2} C_{i2} + \ldots + A_{in} C_{in}$$

Return to

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$

$$\det(A) = A_{11} C_{11} + A_{12} C_{12} + A_{13} C_{13} = 1 \cdot (-8) + 2(10) + 3(4)$$
$$= 24$$

- expansion around row 1

- expansion around row 3:

$$\det(A) = (+2) \begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} - (0) \overset{=0}{\begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix}} + 4 \overset{=0}{\begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix}}$$
$$= 2(6 - (-6)) = 24$$

- row operations

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow[\text{no change}]{r_2 := r_2 + r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow[\text{no change}]{r_1 := r_1 - \frac{1}{2} r_3} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 6 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow[(-1)]{r_3 \leftrightarrow r_1} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 6 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[(-1)]{r_2 \leftrightarrow 3} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix} = B$$

$$\det(B) = 24$$
$$= (-1)^2 \det(A)$$
$$= \det(A)$$

Ex

$$A = \begin{bmatrix} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 8 & 2 & -1 & 0 \\ 1 & 0 & 3 & 4 \end{bmatrix}$$

$$|A| = (-2)\begin{vmatrix} 3 & -1 & 1 \\ 8 & 2 & 0 \\ 1 & 0 & 4 \end{vmatrix} = (-2)\left((1)\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + (4)\begin{vmatrix} 3 & -1 \\ 8 & 2 \end{vmatrix}\right)$$

$$= -2\left(-2 + 4\left(6 - (-8)\right)\right) = -2(54) = -108$$

- Produce the formula for the determinant using row operations.

- If 2×2 determinant requies computatin of 2 terms, then 3×3 requires $3\cdot2=6$ terms
  So 4×4 needs $4\cdot6 = 4\cdot3\cdot2\cdot1 = 4!$ terms.