$\boxed{\text{Putting It Together}}$

$\boxed{\text{eigenvalues/vectors}}$

?

$\boxed{\text{solving a system of linear differential equations}}$

Motivating Problem: Solving a system of linear, first-order differential equations.

Ex | Solve $\boxed{\begin{aligned} \frac{dv}{dt} &= v(t) - w(t) \\ \frac{dw}{dt} &= 2v(t) + 4w(t) \end{aligned}}$ $\boxed{\begin{aligned} v(0) &= 40 \\ w(0) &= 10 \end{aligned}}$

Solution: $v(t) = 90e^{2t} - 50e^{3t}$

$w(t) = -90e^{2t} + 100e^{3t}$

Connection: $u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$ $\boxed{u(0)} = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$

$\boxed{\frac{du}{dt} = Au}$ $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

① Write (formulate) the system as a matrix vector product, with some initial conditions

(2) Use your CalcI knowledge:

Q | Find $y = f(x)$ such that $\frac{dy}{dx} = ay$ and $y(0) = C$

A | $y = Ce^{ax}$

Check $\frac{d}{dx}\left[Ce^{ax}\right] = Ce^{ax} \cdot a = a\left(Ce^{ax}\right) = ay$ ✓

and $y(0) = Ce^{a \cdot 0} = C$ ✓

Generalize to $u(t)$.

Conjecture: $u(t) = Ce^{at}$

called a pure exponential solution

$$u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} e^{at} = \begin{bmatrix} v(0)\, e^{at} \\ w(0)\, e^{at} \end{bmatrix}$$

Consequence:

$$Au = \frac{du}{dt} = \begin{bmatrix} v(0)\, e^{at} \cdot a \\ w(0)\, e^{at} \cdot a \end{bmatrix} = a \begin{bmatrix} v(0)\, e^{at} \\ w(0)\, e^{at} \end{bmatrix} = \underset{\#}{au}$$

So, we want u so that $Au = au$

So, $a = \lambda$, an eigenvalue of A.

and u is an eigenvector of A.

Let's proceed w/ step 2 and find the pure exponential solutions for our example.

Ex $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Find eigenvalues and eigenvectors.

Find eigenvalues

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix}\right) = (1-\lambda)(4-\lambda) - (-1)(2) = \lambda^2 - 5\lambda + 6$$

$$= (\lambda-2)(\lambda-3) = 0. \quad \boxed{\text{So } \lambda_1 = 2,\ \lambda_2 = 3.}$$

Find associated eigenvectors.

$\lambda_1 = 2$: $\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ or $x_1 + x_2 = 0$. Pick $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 3$: $\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ or $x_1 + \frac{1}{2}x_2 = 0$. Pick $x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Pure Exponential Solutions

$$u_1 = Ce^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} Ce^{2t} \\ -Ce^{2t} \end{bmatrix}, \quad u_2 = De^{3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} De^{3t} \\ -2De^{3t} \end{bmatrix}$$

It's easy to check that u_1 and u_2 satisfy $\frac{du}{dt} = Au$. (See next page →)

Check that u_1 and u_2 are solutions to the system $\frac{du}{dt} = Au$.

- $$\frac{du_1}{dt} = \frac{d}{dt}\left(\begin{bmatrix} Ce^{2t} \\ -Ce^{2t} \end{bmatrix}\right) = 2\begin{bmatrix} Ce^{2t} \\ -Ce^{2t} \end{bmatrix} = 2Ce^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Au_1 = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} Ce^{2t} \\ -Ce^{2t} \end{bmatrix} = \begin{bmatrix} 2Ce^{2t} \\ -2Ce^{2t} \end{bmatrix} = 2Ce^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

equal ✓

- $$\frac{du_2}{dt} = \frac{d}{dt}\left(\begin{bmatrix} De^{3t} \\ -2De^{3t} \end{bmatrix}\right) = \begin{bmatrix} 3De^{3t} \\ -6De^{3t} \end{bmatrix} = 3De^{3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$Au_2 = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} De^{3t} \\ -2De^{3t} \end{bmatrix} = \begin{bmatrix} 3De^{3t} \\ -6De^{3t} \end{bmatrix} = 3De^{3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

equal ✓

(3) Use your knowledge of linear functions.

Since $u_1 = Ce^{2t}\begin{bmatrix}1\\-1\end{bmatrix}$ and $u_2 = De^{3t}\begin{bmatrix}1\\-2\end{bmatrix}$

satisfy $\frac{du}{dt} = Au$, then any linear combination of u_1 and u_2 will also be a solution.

So $u(t) = Ce^{2t}\begin{bmatrix}1\\-1\end{bmatrix} + De^{3t}\begin{bmatrix}1\\-2\end{bmatrix}$ is a solution to $\frac{du}{dt} = Au$.

Now find C and D to satisfy $u(0) = \begin{bmatrix}40\\10\end{bmatrix}$.

At $t=0$,

$$\begin{bmatrix}40\\10\end{bmatrix} = u(0) = C\begin{bmatrix}1\\-1\end{bmatrix} + D\begin{bmatrix}1\\-2\end{bmatrix} \quad \text{or} \quad \begin{aligned} C + D &= 40 \\ -C - 2D &= 10. \end{aligned}$$

Solve: $D = -50$, $C = 90$

Answer: $u(t) = 90e^{2t}\begin{bmatrix}1\\-1\end{bmatrix} - 50e^{3t}\begin{bmatrix}1\\-2\end{bmatrix}$ or

$$\left.\begin{aligned} v(t) &= 90e^{2t} - 50e^{3t} \\ w(t) &= -90e^{2t} + 100e^{3t} \end{aligned}\right]$$

We already checked that this is correct!

Nutshell

1. Transform the system of differential equations to matrix-vector form, obtaining a matrix A of coefficients.
2. Find the eigenvalues and associated eigen-vectors of A.
3. Use the eigenvalues/vectors to obtain pure exponential solutions to the diffy. equ.
4. Solve for a solution to the system with initial conditions using a linear combination of pure exponential solutions.