- Recall: If A is tall $m\times n$ matrix $\left(A = \begin{bmatrix} & \end{bmatrix} \; m>n \right)$

  and A has a left inverse, $\left(CA = I_n \right)$

  then the columns of A are linearly independent.

  [why? b/c $Ax = 0 \Rightarrow CAx = C0$

  $\Rightarrow$ $x=0$ is the only solution.]

- Jill asserted: The converse is also true.

  If columns of A are linearly independent, then A is left-invertible.

  [why? By constructing the "pseudoinverse" of A. Specifically, $\boxed{(A^TA)^{-1}A^T}$ is a left-inverse of A.

  Notation: pseudo inverse is $A^\dagger$]

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix}$.

- Find the pseudo inverse of A.
- Find an inverse by inspection.

So $A^T = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$,

So $A^T A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$, $(A^T A)^{-1} = \left(\frac{1}{10-1}\right) \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} =$

$= \begin{bmatrix} \frac{2}{9} & \frac{-1}{9} \\ \frac{-1}{9} & \frac{5}{9} \end{bmatrix}$; $(A^T A)^{-1} A^T = \begin{bmatrix} \frac{2}{9} & \frac{-1}{9} \\ \frac{-1}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{4}{9} & \frac{-2}{9} & \frac{-5}{9} \end{bmatrix}$;

check!

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{4}{9} & \frac{-2}{9} & \frac{-5}{9} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{9}+\frac{8}{9} & \frac{1}{9}-\frac{1}{9} \\ \frac{4}{9}-\frac{4}{9} & \frac{4}{9}+\frac{5}{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$