1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 2 & -1 & -1 & 2 \end{bmatrix}$, find

   (a) AB

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 2 & -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1+6 & 1+4-3 & -1+2-3 & -2+6 \\ 4+12 & 4+10-6 & -4+5-6 & -5+12 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -2 & 4 \\ 16 & 8 & -5 & 7 \end{bmatrix}$$

   (b) BA not defined.

lesson: AB may not always be defined!

2. For $A = \begin{bmatrix} 2 & 5 \\ \pi & \sqrt{2} \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$, find

   (a) AB **by first writing out the calculation consistently in detail** and then completing the calculation. (See the process started for you...)

$$AB = \begin{bmatrix} 2 & 5 \\ \pi & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 5 \cdot 3 & 2 \cdot (-1) + 5 \cdot 2 \\ \pi \cdot 1 + \sqrt{2}\, 3 & \pi (-1) + \sqrt{2} \cdot 2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 3\sqrt{2}+\pi & 2\sqrt{2}-\pi \end{bmatrix}$$

- You can see copies of A in the product.
- You can see columns of B in the product.

Epiphany $\longrightarrow$ $AB = \begin{bmatrix} | & | \\ A \cdot col_1(B) & A \cdot col_2(B) \\ | & | \end{bmatrix}$

matrix·vector product

   (b) BA (no extra restrictions)

$$BA = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ \pi & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2-\pi & 5-\sqrt{2} \\ 6+2\pi & 15+2\sqrt{2} \end{bmatrix}$$

Lesson: Even when both AB and BA are defined, they may NOT be equal. That is, matrix multiplication is NOT commutative.

$$A = \begin{bmatrix} 2 & 5 \\ \pi & \sqrt{2} \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$$

(c) $(AB)^T$ (Use part (a)....)

$$AB = \begin{bmatrix} 17 & 8 \\ 3\sqrt{2}+\pi & 2\sqrt{2}-\pi \end{bmatrix}; \text{ so } (AB)^T = \begin{bmatrix} 17 & 3\sqrt{2}+\pi \\ 8 & 2\sqrt{2}-\pi \end{bmatrix}$$

(d) $B^TA^T = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & \pi \\ 5 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2+15 & \pi+3\sqrt{2} \\ -2+10 & -\pi+2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 17 & 3\sqrt{2}+\pi \\ 8 & 2\sqrt{2}-\pi \end{bmatrix}$

Lesson: $(AB)^T = B^T A^T$

(e) $A^2 = \begin{bmatrix} 2 & \pi \\ 5 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & \pi \\ 5 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 4+5\pi & (2+\sqrt{2})\pi \\ 10+5\sqrt{2} & 5\pi+2 \end{bmatrix}$

Lesson: $A^2 = AA$.

3. Find each product below.

(a) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ -8 & -14 \\ 12 & 21 \end{bmatrix}$

Can be framed as a vector outer-product.

$(3 \times 1)(1 \times 2)$ gives 3×2 product!

(b) $\begin{bmatrix} \pi & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} \pi & 2\pi & 3\pi \\ 4\sqrt{2} & 5\sqrt{2} & 6\sqrt{2} \end{bmatrix}$

a diagonal matrix multiplied on the LEFT multiplies each row by its diagonal entries

What if the diagonal matrix is on the left?

(c) $I_2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Identity matrices are multiplicative identities

$AI_n = A$; $I_n A = A$.

4. Suppose A and B are both 2×2 matrices. Let $C = \begin{bmatrix} A & I \\ 0 & B \end{bmatrix}$ be matrix defined in terms of blocks where I is the identity matrix and 0 is the zero matrix.

(a) What are the dimensions of I? Of 0? Of C?

$$C = \begin{bmatrix} \boxed{A} & I \\ 0 & \boxed{B} \end{bmatrix}$$

2 2 2 2

$I = I_2$, 0 is 2×2 matrix of all zeros

C is 4×4.

(b) Find C^2.

$$C^2 = \begin{bmatrix} A & I_2 \\ 0 & B \end{bmatrix} \begin{bmatrix} A & I_2 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A^2 & A+B \\ 0 & B^2 \end{bmatrix}$$

5. Observations/Rules

- matrix multiplication is not commutative : AB may NOT be BA.
- I_n acts like a multiplicative identity
- $(AB)^T = B^T A^T$

[Additional Properties:

- $(AB)C = A(BC)$
- $A(B+C) = AB + AC$
- $\alpha(AB) = (\alpha A)B = A(\alpha B)$]

$*$ So $(\alpha A + B)(\beta C + D) = \alpha\beta AC + \alpha AD + \beta BC + BD$

- $$\underset{(m\times p)(p\times n)}{AB} = A \begin{bmatrix} | & | & & | \\ col_1(B) & col_2(B) & \cdots & col_n(B) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\,col_1(B) & A\,col_2(B) & \cdots & A\,col_n(B) \\ | & | & & | \end{bmatrix}$$

Your book calls this "the column interpretation of matrix·matrix multiplication."

- If $C = AB$, then $C_{ij} = \underbrace{row_i(A)\, col_j(B)}_{\text{vector inner product}}$

- A "row interpretation"?

$$AB = \begin{bmatrix} \text{—— } row_1(A) \text{ ——} \\ \text{—— } row_2(A) \text{ ——} \\ \vdots \\ \text{—— } row_m(A) \text{ ——} \end{bmatrix} B = \begin{bmatrix} \text{—— } row_1(A)\cdot B \text{ ——} \\ \text{—— } row_2(A)\cdot B \text{ ——} \\ \vdots \\ \text{—— } row_m(A)\cdot B \text{ ——} \end{bmatrix}$$

That is, if $C = AB$, then $row_i(C) = row_i(A) \cdot B$

$$= [\text{———}] \begin{bmatrix} | & | & \cdots & | \end{bmatrix}$$