1. *Definition:* The matrix C is called the *left inverse* of matrix A if

   In this case, we call the matrix A *left-invertible*.

2. Suppose A is an $m \times n$ matrix. What must be the dimensions of I and C in the definition above?

$$\underset{n \times m}{\begin{bmatrix} & C & \end{bmatrix}} \underset{m \times n}{\begin{bmatrix} & A & \end{bmatrix}} = I_n$$

$n \times n$

3
2
1

3. Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$

   (a) Show that A is left-invertible by finding a left inverse C of A.

$$\underset{C}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}} = \underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \checkmark$$

An inverse for A^T?

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   (b) Is the left inverse, C, unique? No

$$\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$ will also work

   (c) Is A right-invertible? Justify your conclusion. No

$$\underset{\substack{3 \times 2 \\ A}}{\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}} \underset{\substack{2 \times 3 \\ C}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}} = \underset{\substack{3 \times 3 \\ I_3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

To get the first row of I_3, the first row of C is forced.
The same for the second row.
Now the third row is wrong.

4. Give an example of a matrix A that has at least one non-zero entry but for which no left-inverse or right-inverse exists. Give some justification that you are correct.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ For any CA, the last column will be all zeros

For any AC, the last row will be all zeros.

5. Let $S = \begin{cases} 4x + 7y = 10 \\ 2x + 6y = 20 \end{cases}$.

(a) Write S as a matrix equation $Ax = b$.

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$CA = \begin{bmatrix} 2.4 - 1.4 & 4.2 - 4.2 \\ -0.8 + 0.8 & -1.4 + 2.4 \end{bmatrix} = I_2 \checkmark$$

(b) Check that $C = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$ is the left inverse of $A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$ and use C to solve the system S.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 6 - 14 \\ -2 + 8 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}$$

check

$-32 + 42 = 10 \checkmark$

$-16 + 36 = 20 \checkmark$

(c) Show that C is also a right-inverse of A.

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 2.4 - 1.4 & 0 \\ 0 & -1.4 + 2.4 \end{bmatrix} = I_2$$

(d) Can you find an inverse of A^T?

$$\underset{C^T}{\begin{bmatrix} 0.6 & -0.2 \\ -0.7 & 0.4 \end{bmatrix}} \underset{A^T}{\begin{bmatrix} 4 & 2 \\ 7 & 6 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6. Let $a_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}$, $a_2 = \begin{bmatrix}1\\1\\3\end{bmatrix}$ and $a_3 = \begin{bmatrix}2\\1\\4\end{bmatrix}$.

(a) Write the system of equations you need to solve to show a_1, a_2, and a_3 are linearly independent. (Note that you are not asked to solve this system.)

Solve

$\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0$

$$\begin{aligned} \beta_1 + \beta_2 + 2\beta_3 &= 0 \\ \beta_1 + \beta_2 + \beta_3 &= 0 \\ 2\beta_1 + 3\beta_2 + 4\beta_3 &= 0 \end{aligned}$$

(b) Write the system of equations from part (a) as a matrix equation $Ax = b$.

$$\begin{bmatrix}1 & 1 & 2\\1 & 1 & 1\\2 & 3 & 4\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}| & | & |\\a_1 & a_2 & a_3\\| & | & |\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

(c) Check that the matrix $C = \begin{bmatrix}1 & 2 & -1\\-2 & 0 & 1\\1 & -1 & 0\end{bmatrix}$ is a left inverse of $A = \begin{bmatrix}1 & 1 & 2\\1 & 1 & 1\\2 & 3 & 4\end{bmatrix}$ and use C to solve the matrix equation in part (b).

Since $Ax = b$,

$CAx = Cb$. So

$x = Cb$.

$$x = \begin{bmatrix}1 & 2 & -1\\-2 & 0 & 1\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

So only 1 solution!

(d) What conclusions can you draw about a matrix A if it has a left inverse C?

If A has a left inverse, its columns are linearly independent

If A has a right inverse, its rows are linearly independent.

7. Observations

- Left/right inverses may not be unique.
  - However, IF a matrix A is left- and right-invertible, then these are unique and equal.

    If $XA=I$ and $AY=I$, then

    $$X = XI = X(AY) = (XA)Y = IY = Y.$$

    If X and $\tilde{X}$ both have the property that $XA = I = \tilde{X}A$. Then both are equal to Y and thus each other.
  - So A must be square.
  - Defn: A square matrix A is called invertible if A has an inverse A^{-1}.
    [ie there is a matrix A^{-1} so that $A^{-1}A = AA^{-1} = I_n$]
    Otherwise called singular ($\equiv$ no inverse)
- If C is a left inverse to A, then C^T is a right inverse to A.
  [i.e. We know $(XY)^T = Y^T X^T$. So
  if $CA = I$, then $(CA)^T = I^T = I$.
  But now $I = (CA)^T = A^T C^T$. ✓]
- Given a system of equations in vector form: $Ax = b$.
  If A is invertible, then $x = A^{-1}b$.