NOTES: SOLUTIONS TO SYSTEMS OF LINEAR EQUATIONS AND GAUSSIAN ELIMINATION

1. Systems of Linear Equations

$$S_1 = \begin{cases} x + \ \ y = 4 \\ x + 2y = -1 \end{cases} \qquad S_2 = \begin{cases} x + 2y = 1 \\ x + 2y = 2 \end{cases} \qquad S_3 = \begin{cases} x_1 + \ \ x_2 + 3x_3 = 5 \\ x_1 + 2x_2 + 4x_3 = 6 \end{cases}$$

Encoding

Matrix-vector Multiplication

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} y = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Augmented Matrix

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 3 & 5 \\ 1 & 2 & 4 & 6 \end{bmatrix}$$

2. A **solution** to a system of equations is a set of values (numbers) for the variables so that *all* equations in the system are true.

$$S_1 = \begin{cases} x + \ \ y = 4 \\ x + 2y = -1 \end{cases} \qquad S_2 = \begin{cases} x + 2y = 1 \\ x + 2y = 2 \end{cases} \qquad S_3 = \begin{cases} x_1 + \ \ x_2 + 3x_3 = 5 \\ x_1 + 2x_2 + 4x_3 = 6 \end{cases}$$

A solution

$x = 9,\ y = -5$	none	$x_1 = 2, x_2 = 0, x_3 = 1$ $x_1 = 4, x_2 = 1, x_3 = 0$

3. **To solve** as system of equations is to find *all possible solutions*.

   - For system $S_1 : x = 9,\ y = -5$ is the only solution.
   - For system S_2 : no solutions
   - For system S_3 : an infinite number of solutions. Specifically, the solution set can be written

$$\left\{ \begin{bmatrix} 4 - 2x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix} \ : \ x_3 \text{ is any real number} \right\}$$

4. **Observation:** The operations below do not change the solutions to a system of equations.

(a) Reordering the equations

$$S_1 = \begin{cases} x + y = 4 \\ x + 2y = -1 \end{cases} \qquad S_1' = \begin{cases} x + 2y = -1 \\ x + y = 4 \end{cases}$$

(b) Multiplying an equation by a constant

$$S_1 = \begin{cases} x + y = 4 \\ x + 2y = -1 \end{cases} \qquad S_1' = \begin{cases} \pi x + \pi y = 4\pi \\ x + 2y = -1 \end{cases}$$

(c) Adding a multiple of one equation to another

$$S_1 = \begin{cases} x + y = 4 \\ x + 2y = -1 \end{cases} \qquad S_1' = \begin{cases} x + y = 4 \\ 3x + 4y = 7 \end{cases}$$

NOTE: The equation $\boxed{3x + 4y = 7}$ is obtained by $\boxed{\text{(equ. 2)} + 2\text{(equ. 1)}}$ or, equivalently, $\boxed{(x + 2y) + 2(x + y) = -1 + 2(4).}$

5. **Observation:** To operations in item #4 above can be described by **row operations** performed on the augmented matrix.

These are called **elementary row operations**.

Encode $\boxed{S_1 = \begin{cases} x + y = 4 \\ x + 2y = -1 \end{cases}}$ as $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -1 \end{bmatrix}$

(a) Reorder rows $\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -1 \end{bmatrix}$ $\quad r_1 \leftrightarrow r_2 \quad$ $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$

(b) Multiply a row by a constant $\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -1 \end{bmatrix}$ $\quad \pi * r_1 \to r_1 \quad$ $\begin{bmatrix} \pi & \pi & 4\pi \\ 1 & 2 & -1 \end{bmatrix}$

(c) Adding a multiple of one row to another $\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -1 \end{bmatrix}$ $\quad 2r_1 + r_2 \to r_2 \quad$ $\begin{bmatrix} 1 & 1 & 4 \\ 3 & 4 & 7 \end{bmatrix}$

6. Gaussian Elimination

Solve a system of equations by

(1) encoding the system as an augmented matrix,

(2) repeatedly use elementary row operations to construct the matrix of an **equivalent** system that is **simple** to solve, then

(3) solve the new simple system.