WORKSHEET: GAUSSIAN ELIMINATION (A FIRST LOOK)

1. Use Gaussian elimination to solve the system $S_1 = \begin{cases} x + y = 1 \\ 2y - z = -4 \\ x + y + z = 4 \end{cases}$ $\xrightarrow{\text{encode}}$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & -1 & -4 \\ 1 & 1 & 1 & 4 \end{bmatrix} \xrightarrow{r_3 - r_1 \to r_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2} r_2 \to r_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1/2 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

row echelon form

$$\xrightarrow{r_1 - r_2 \to r_2} \begin{bmatrix} 1 & 0 & 1/2 & 3 \\ 0 & 1 & -1/2 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[r_2 + \frac{1}{2} r_3 \to r_2]{r_1 - \frac{1}{2} r_3 \to r_1} \begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[\text{to equations}]{\text{return}}$$

$$\boxed{\begin{aligned} x &= 3/2 \\ y &= -1/2 \\ z &= 3 \end{aligned}}$$

reduced row echelon form

2. Use Gaussian elimination to solve the system $S_2 = \begin{cases} x - 5y + z = 2 \\ x - 4y + z = 2 \\ 2x + z = 5 \end{cases}$

$$\begin{bmatrix} 1 & -5 & 1 & 2 \\ 1 & -4 & 1 & 2 \\ 2 & 0 & 1 & 5 \end{bmatrix} \xrightarrow[r_3 - 2r_1 \to r_3]{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & -5 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 10 & -1 & 1 \end{bmatrix} \xrightarrow[r_1 + 5r_2 \to r_1]{r_3 - 10r_2 \to r_3} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$-2 \quad +10 \quad -2 \quad -4$

$$\xrightarrow{-1 r_3 \to r_3} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_1 - r_3 \to r_1} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

rref

Solution: $x = 3,\ y = 0,\ z = -1$

3. Use Gaussian elimination to solve the system $S_3 = \begin{cases} x_1 + x_2 + 3x_3 = 5 \\ x_1 + 2x_2 + 4x_3 = 6 \end{cases}$

$$\begin{bmatrix} \boxed{1} & 1 & 3 & 5 \\ 1 & 2 & 4 & 6 \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & \boxed{1} & 1 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2 \to r_1} \boxed{\begin{bmatrix} \boxed{1} & 0 & 2 & 4 \\ 0 & \boxed{1} & 1 & 1 \end{bmatrix}}$$

rref

$$\begin{aligned} x_1 + 2x_3 &= 4 \\ x_2 + x_3 &= 1 \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 &= 4 - 2x_3 \\ x_2 &= 1 - x_3 \end{aligned}$$

Solution $\left\{ \begin{bmatrix} 4 - 2x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix} : x_3 \text{ any real number} \right\}$

$\boxed{\text{reduced row echelon form of a matrix}}$

- any row(s) of all zeros at bottom
- any nonzero row leads with a $\boxed{1}$.
  (i.e. left-most entry is $\boxed{1}$)
- any column with a leading $\boxed{1}$ has all other entries zero
- leading $\boxed{1}$ in a row is to the right of the leading $\boxed{1}$ in row above.