Notes: Reduced Row-Echelon Form and Gaussian Elimination with Partial Pivoting

1. Examples from Monday

$$S_1 = \begin{cases} x + y = 1 \\ 2y - z = -4 \\ x + y + z = 4 \end{cases} \qquad \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 2 & -1 & -4 \\ 1 & 1 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$S_3 = \begin{cases} x_1 + x_2 + 3x_3 = 5 \\ x_1 + 2x_2 + 4x_3 = 6 \end{cases} \qquad \begin{bmatrix} 1 & 1 & 3 & 5 \\ 1 & 2 & 4 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

2. Reduced Row Echelon Form
   (a) Rows of all zeros are at the bottom.
   (b) Every row with nonzero entries has a 1 in the left-most entry. (Called the **leading one** or **pivot**)
   (c) If a row has a leading 1, it is to the right of all leading 1's in the rows above.
   (d) Each column with a leading 1 has zeros in all other entries.

3. Example A

$$S_4 = \begin{cases} v + 2w + y = -1 \\ 2v + 4w + x + y = 0 \\ -v - 2w + x - 2y + 2z = 11 \\ v + 2w + x + z = 5 \end{cases} \qquad \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 2 & 4 & 1 & 1 & 0 & 0 \\ -1 & -2 & 1 & -2 & 2 & 11 \\ 1 & 2 & 1 & 0 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Example B: Solve $\begin{cases} 2w + 4x - 2y - 2z = -4 \\ w + 2x + 4y - 3z = 5 \\ -3w - 3x + 8y - 2z = 7 \\ -w + x + 6y - 3z = 7 \end{cases}$.

$$\begin{bmatrix} 2 & 4 & -2 & -2 & -4 \\ 1 & 2 & 4 & -3 & 5 \\ -3 & -3 & 8 & -2 & 7 \\ -1 & 1 & 6 & -3 & 7 \end{bmatrix} r_2 - (1/2)r_1 \to r_2 \qquad r_3 + (3/2)r_1 \to r_3 \qquad r_4 + (1/2)r_1 \to r_4$$

$$\begin{bmatrix} 2 & 4 & -2 & -2 & -4 \\ 0 & 0 & 5 & -2 & 7 \\ 0 & 3 & 5 & -5 & 1 \\ 0 & 3 & 5 & -4 & 5 \end{bmatrix} \quad r_2 \leftrightarrow r_4$$

$$\begin{bmatrix} 2 & 4 & -2 & -2 & -4 \\ 0 & 3 & 5 & -4 & 5 \\ 0 & 3 & 5 & -5 & 1 \\ 0 & 0 & 5 & -2 & 7 \end{bmatrix} \quad r_3 - r_2 \to r_3$$

$$\begin{bmatrix} 2 & 4 & -2 & -2 & -4 \\ 0 & 3 & 5 & -4 & 5 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 5 & -2 & 7 \end{bmatrix} \quad r_4 \leftrightarrow r_3$$

$$\begin{bmatrix} 2 & 4 & -2 & -2 & -4 \\ 0 & 3 & 5 & -4 & 5 \\ 0 & 0 & 5 & -2 & 7 \\ 0 & 0 & 0 & -1 & -4 \end{bmatrix}$$ keep going