1. An example of a system of linear, first-order differential equations.

d.e.
$$\left[\begin{array}{lll} \frac{dv}{dt} = v(t) - w(t) & ① & v(0) = 40 \\ \frac{dw}{dt} = 2v(t) + 4w(t) & ② & w(0) = 10 \end{array}\right]$$
initial conditions

2. A solution: $v(t) = 90e^{2t} - 50e^{3t}, \quad w(t) = -90e^{2t} + 100e^{3t}$.

Are initial conditions correct?

$v(0) = 90 - 50 = 40$ ✓

$w(0) = -90 + 100 = 10$ ✓

Are the diff. eq. correct?

① $\frac{dv}{dt} = 180e^{2t} - 150e^{3t}$ ✓

$v(t) - w(t) = (90e^{2t} - 50e^{3t}) - (-90e^{2t} + 100e^{3t})$

$= 180e^{2t} - 150e^{3t}$ ✓

② $\frac{dw}{dt} = -180e^{2t} + 300e^{3t}$ ✓

$2v(t) + 4w(t) = (180e^{2t} - 100e^{3t}) + (-360e^{2t} + 400e^{3t})$

$= -180e^{2t} + 300e^{3t}$ ✓

3. Let A be an $m \times n$ matrix, then the **null space of** A is the set of all n-vectors x so that $Ax = 0$. It's denoted $N(A)$.

4. Example 1: $A = \begin{bmatrix} 1 & 2 \\ 10 & 20 \end{bmatrix}$

$x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ in $N(A)$ b/c $\begin{bmatrix} 1 & 2 \\ 10 & 20 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $N(A)$ b/c $\begin{bmatrix} 1 & 2 \\ 10 & 20 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 30 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$N(A) = \left\{ \begin{bmatrix} 2a \\ -a \end{bmatrix} : a \text{ any real \#} \right\}$

5. Example 2: $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$ Find $N(B)$. Solve $Bx = 0$ for $\boxed{\text{all}}$ x-vectors

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 5 & 4 & 9 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow \begin{array}{l} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{array} \rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = -x_3 \end{array}$$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$.

Answer: $N(B) = \left\{ \begin{bmatrix} c \\ c \\ -c \end{bmatrix} : c \text{ any real \#} \right\}$

6. Show/prove/give an argument for each of the statements below.

(a) For every matrix A, $N(A) \neq \emptyset$.

n-vector m-vector

0 in N(A) because $A0 = 0$ for all matrices

(b) If A is invertible, then $N(A)$ contains only the zero vector.

If A is invertible, then $Ax=0$ implies $x = A^{-1}0 = 0$.

(c) If the vector x is in $N(A)$, then for any number c, cx is in $N(A)$.

If x in N(A), then $Ax=0$.

So $A(cx) = cAx = c \cdot 0 = 0$.

So cx in N(A)

(d) If both of the vectors x and y are in $N(A)$, then the vector $x + y$ is also in $N(A)$.

If x, y in N(A), then $Ax=0$ and $Ay=0$.

So $A(x+y) = Ax + Ay = 0+0=0$.

(e) If x is in $N(A)$ and the vector z is *not* in $N(A)$, then x and z are linearly independent.

If x and y were linearly dependent, then $y = cx$ for some c. Since x in N(A), cx in N(A) by part (c).

But this is impossible since y is not in N(A).

(f) If the vector a is in $N(A)$ and c is a solution to $Ax = b$, then $c + a$ is also a solution to $Ax = b$.

$A(c+a) = Ac + Aa = b + 0 = b$

(g) If both of the vectors c_1 and c_2 are solutions to $Ax = b$, then there is a vector a in $N(A)$ such that $c_2 = c_1 + a$.

Choose $a = c_2 - c_1$. By this choice $c_2 = c_1 + a$. (!!)

Now $Aa = A(c_2 - c_1) = Ac_2 - Ac_1 = b - b = 0$. So a in N(A). !!

7. Main Principles See notes.