WORKSHEET: VECTOR ALGEBRA, LINEAR AND AFFINE FUNCTIONS

1. Label each of the statements below TRUE or FALSE.

   Let $a, u,$ and v be n-vectors and let α and β be scalars. All are true

   (a) $a^T u = u^T a$

   (b) $\alpha(u+v) = \alpha u + \alpha v$

   (c) $\alpha(a^T u) = (\alpha a)^T u$

   (d) $\alpha(a^T u) = a^T(\alpha u)$

   (e) $a^T(u+v) = a^T u + a^T v$

   (f) $\beta(a^T u) + \beta = \beta(a^T u + 1)$

2. Complete the definition of a *linear vector function*:
   The function $f : \mathbb{R}^n \to \mathbb{R}$ is linear if for every pair of vectors u and v and every pair of scalars α and β,

   $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$

3. Make up two examples of functions $f : \mathbb{R}^3 \to \mathbb{R}$, one that is linear and one that is not linear.

   many examples!

   linear: $f(x) = a^T x$

   not linear: $f(x) = x_1 x_2$

4. Every linear function can be written as $f(x) = a^T x$ for appropriate vector a.

5. The definition of an *affine vector function*:
   The function $f : \mathbb{R}^n \to \mathbb{R}$ is *affine* if for every pair of vectors u and v and every pair of scalars α and β

   $\ast$ $\boxed{f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)}$ provided $\boxed{\alpha + \beta = 1}$

   why? see notes!

6. Every affine function can be written as $f(x) = a^T x + c$ where c is a scalar.

Recall Example 3 from previous worksheet: $f(x) = 4x_1 - x_2 + 2$

So $f(x) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2$

really affine for us!

7. "Linear" Taylor Approximations, $\hat{f}$

$f: \mathbb{R}^n \to \mathbb{R}$ differentiable vector function, z - an n-vector in domain of f.

$$\hat{f}(x) = f(z) + \frac{\partial f}{\partial x_1}(z)(x_1 - z_1) + \frac{\partial f}{\partial x_2}(z)\cdot(x_2 - z_2) + \cdots + \frac{\partial f}{\partial x_n}(z)(x_n - z_n)$$

or

$$\hat{f}(x) = \left(\nabla f(z)\right)^T \cdot (x - z) + f(z)$$

8. Let $f(x) = x_1 e^{-x_2} + x_3$ and $z = (2, 0, 1)$.

(a) Find $\hat{f}(x)$, the ~~linear~~ degree 1 Taylor approximation of f at z.

ingredients:

$f(2,0,1) = 2 \cdot e^0 + 1 = 3$

$\frac{\partial f}{\partial x_1} = e^{-x_2}$, $\frac{\partial f}{\partial x_1}(2,0,1) = 1$

$\frac{\partial f}{\partial x_2} = -x_1 e^{-x_2}$, $\frac{\partial f}{\partial x_2}(2,0,1) = -2$

$\frac{\partial f}{\partial x_3} = 1$, $\frac{\partial f}{\partial x_3}(2,0,1) = 1$.

$$\hat{f}(x) = 3 + 1(x_1 - 2) + (-2)(x_2 - 0) + 1(x_3 - 1)$$

$$= 3 + (x_1 - 2) - 2x_2 + (x_3 - 1)$$

(b) Find $f(2.1, 0.1, 0.9)$ and $\hat{f}(2.1, 0.1, 0.9)$.

$$f(2.1, 0.1, 0.9) = 2.1(e^{-0.1}) + 0.9 = 2.8001585\ldots$$

$$\hat{f}(2.1, 0.1, 0.9) = 3 + (0.1) - 2(0.1) + (-0.1)$$

$$= 3.1 - 0.3 = 2.9$$

close