WORKSHEET: FACT A, BASIS, AND ORTHONORMAL VECTORS

1. Fact A: (independence-dimension inequality)

   or equivalently.

   [A linearly independent set of n-vectors can have at most n elements.

   Any set of $n+1$ or more n-vectors must be linearly dependent.

   See Lin. Indep. Worksheet #s 1c, 2e

   So if we want a set of vectors, all of which are 21-vectors, and we want them to be linearly independent, our set can have $1, 2, 3, \ldots, 20$, or 21 vectors. But not $22, 23, \ldots$

2. Definition: A *basis* is

   a set of n linearly independent n-vectors.

   Pick an arbitrary 3-vector... say $(-\sqrt{2}, 17, \pi) = b$

   Write b wrt B_1 and B_2

   $(-\sqrt{2}, 17, \pi) = -\sqrt{2}e_1 + 17e_2 + \pi e_3$

3. Give three distinct examples of bases when

   (a) $n = 2$ (We need two linearly independent 2-vectors.)

   $B_1 = \{(1,0),(0,1)\}$

   $B_2 = \{(2,0),(0,100)\}$

   $B_3 = \{(1,1),(-5,17)\}$

   $(-\sqrt{2}, 17, \pi) = (-\sqrt{2}-\pi)\begin{bmatrix}1\\0\\0\end{bmatrix} + (17-\pi)\begin{bmatrix}0\\1\\0\end{bmatrix} + \pi\begin{bmatrix}1\\1\\1\end{bmatrix}$

   (b) $n = 3$ (We need three linearly independent 3-vectors.)

   $B_1 = \{(1,0,0),(0,1,0),(0,0,1)\} = \{e_1, e_2, e_3\}$

   $B_2 = \{(1,0,0),(0,1,0),(1,1,1)\}$

   $B_3 = \{(1,2,3),(-1,2,-1),(0,0,1)\}$

   $B_4 = \{(1,1,1),(-1,1,0),(0,1,1)\}$

   From prev. worksheet.

4. Fact B: (pg 92) If $a_1, a_2, \ldots, a_n$ is a basis of n-vectors, then EVERY n-vector b can be written as a linear combination of the a_i's AND this representation is unique.

   why?

   or $a_1, a_2, \ldots, a_n$ basis $\Rightarrow$ unique $\beta_1, \beta_2, \ldots, \beta_n$ so that

   $b = \beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_n a_n$ no matter how you choose b.

Suppose $a_1, a_2, \ldots, a_n$ forms a basis So a_i's are $\underline{n}$-vectors

and let b be some arbitrary n-vector.

How do we know there are constants $c_1, c_2, \ldots, c_n$ so that

$$b = c_1 a_1 + c_2 a_2 + \ldots + c_n a_n ?$$

Ans: Fact A $\Rightarrow$ $b, a_1, a_2, \ldots, a_n$ are linearly dependent.

So $0_n = \beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_n a_n + \beta_{n+1} b$ where the

b_i's are not all zero. If $\beta_{n+1} = 0$, then a_i's are linearly dependent which is impossible. So β_{n+1} is not zero

and we can solve for b:

$$b = \frac{1}{\beta_{n+1}} \left(\beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_n a_n \right) \checkmark$$

How do we know the c_i's are unique?

If $b = \sum_{i=1}^{n} c_i a_i$ and $b = \sum_{i=1}^{n} d_i a_i$ and $c_i \neq d_i \ \forall i$,

then $0 = \sum (c_i - d_i) a_i$. $\Rightarrow\Leftarrow$

5. A set of n-vectors $a_1, a_2, \cdots, a_k$ is called *orthogonal* if $a_i \perp a_j$ for all $i \neq j$.

or $a_i^T a_j = 0$ for all $i \neq j$.

6. Examples

$B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$B_2 = \{\underset{v_1}{(1,1,1)}, \underset{v_2}{(\tfrac{1}{2},\tfrac{1}{2},-1)}, \underset{v_3}{(-1,1,0)}\}$$

I'm checking

$v_1^T v_2 = \frac{1}{2} + \frac{1}{2} - 1 = 0$

$v_1^T v_3 = -1 + 1 = 0$

$v_2^T v_3 = -\frac{1}{2} + \frac{1}{2} + 0 = 0$

7. A vector a is called *normal* if $\|a\| = 1$.

8. Examples

$$a = \frac{(\frac{1}{2},\frac{1}{2},-1)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}} = \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{2},\frac{1}{2},-1\right) = \left(\frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{2\sqrt{3}}, \frac{-\sqrt{2}}{\sqrt{3}}\right)$$

$a = (1,0,0)$ ✓

$a = (\cancel{1,1,1})$ but

$$a = \frac{(1,1,1)}{\|(1,1,1)\|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) ✓$$

9. A set of n-vectors $a_1, a_2, \cdots, a_k$ is called *orthonormal* if

- the a_i's are orthogonal
and
- a_i's are all normal.

10. Examples

$B_1 = \{e_1, e_2, e_3\}$

$$\tilde{B}_2 = \left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{2},\frac{1}{2},-1\right), \frac{1}{\sqrt{2}}(1,-1,0)\right\}$$

11. Suppose $a_1, a_2, a_3,$ and a_4 is a set of orthonormal 32-vectors. Further, suppose that $\beta_1, \beta_2, \beta_3$ and β_4 have the property that

$$\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 + \beta_4 a_4 = 0_{32}.$$

(a) Find $a_3^T(\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 + \beta_4 a_4)$.

$= \beta_1 a_3^T a_1 + \beta_2 a_3^T a_2 + \beta_3 a_3^T a_3 + \beta_4 a_3^T a_4$ b/c $a_3 \perp a_1$, $a_3 \perp a_2$, $a_3 \perp a_4$

$= 0 + 0 + \beta_3 \|a_3\|^2 + 0$ b/c $\|a_3\| = 1$

$= \beta_3$

(b) Find $a_3^T 0_{32}$.

$= 0$ (always!)

(c) What can you conclude about β_3? About β_i for $i = 1, 2, 4$?

$\beta_3 = 0$.

So $\beta_1 = \beta_2 = \beta_4 = 0$ since we could replace a_3 with $a_1, a_2,$ or a_4 in the calculations!

(d) What can you conclude about the set $a_1, a_2, a_3,$ and a_4? About *any* set of orthonormal vectors?

Any set of orthonormal vectors are linearly independent.

Any set of n orthonormal n-vectors forms a basis.