Definitions:

The set of vectors, $a_1, a_2, \cdots, a_k$, is **linearly dependent** if the are constants $\beta_1, \beta_2, \dots, \beta_k$ so that

* $\underbrace{\beta_1 a_1 + \beta_2 a_2 + \dots + \beta_k a_k}_{\text{a linear combination of the } a_i\text{'s}} = 0$ ← vector

where the β_i's are NOT all 0.

(It should be clear that we can make this * eq. true by selecting $\beta_1 = \beta_2 = \dots = \beta_k = 0$.)

The set of vectors, $a_1, a_2, \cdots, a_k$, is **linearly independent** if the only solution to

* $\beta_1 a_1 + \beta_2 a_2 + \dots + \beta_k a_k = 0$

is when $\beta_1 = \beta_2 = \dots = \beta_k = 0$.

(So the trivial choice for β_i's is the only solution to *.)

β_i's not all zero

Ex: $a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Claim: $2 \cdot a_1 + (-1) a_2 + (-1) a_3 = 0_4$

check: $\begin{bmatrix} 2 \\ 4 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

$a_3 = 2a_1 - a_2$ or

$a_1 = \frac{1}{2} a_2 + \frac{1}{2} a_3$

Ex: $a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

N.t.s. $\beta_1 = \beta_2 = \beta_3 = 0$ always where

$\beta_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \beta_3 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

or, equivalently

$$\begin{aligned} \beta_1 + 2\beta_2 + \beta_3 &= 0 \\ 2\beta_1 + \beta_3 &= 0 \\ -\beta_1 + 3\beta_2 &= 0 \\ \beta_1 &= 0 \end{aligned}$$

plug in

$\beta_3 = 0$

$3\beta_2 = 0, \beta_2 = 0$

So, $\beta_1 = \beta_2 = \beta_3 = 0$

Equivalent Definitions:

The set of vectors, $a_1, a_2, \cdots, a_k$, is **linearly dependent** if for some i, a_i can be written as a linear combination of the other a_i's.

$\boxed{a_i} = \underbrace{\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_{i-1} a_{i-1} + \alpha_{i+1} a_{i+1} + \dots + \alpha_n a_n}_{\text{the others.}}$

The set of vectors, $a_1, a_2, \cdots, a_k$, is **linearly independent** if no a_i can be written as a linear combination of the other a_i's.

1. Let $a_1 = (1, 2) \in \mathbb{R}^2$.

(a) Find at least 3 different choices for a vector $a_2 \in \mathbb{R}^2$ such that the set a_1, a_2 is linearly **dependent**.

- $a_2 = (1,2)$. So $a_1 - a_2 = (0,0)$
- $a_2 = (-1,-2)$. So $a_1 + a_2 = (0,0)$
- $a_2 = (3,6)$. So $a_1 + (-\frac{1}{3}) a_2 = (0,0)$

Any scalar multiple of a_1 will work.

(b) Find a choice for vector $a_2 \in \mathbb{R}^2$ such that a_1, a_2 are linearly **independent** and demonstrate that you are correct or explain why this is not possible.

Say $a_2 = (1,0)$. N.t.s $\beta_2 = \beta_1 = 0$ for all $\beta_1 a_1 + \beta_2 a_2 = 0$.

equivalently: $\beta_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{aligned} \beta_1 + \beta_2 &= 0 \text{ and} \\ 2\beta_1 &= 0. \end{aligned}$

Since $\beta_1 = 0$, it follows that $\beta_2 = 0$.

2 equ, 3 unknowns

(c) Find a choices for vector $a_2, a_3 \in \mathbb{R}^2$ such that a_1, a_2, a_3 are linearly **independent** and demonstrate that you are correct or explain why this is not possible.

$$\beta_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta_3 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{aligned} \beta_1 + \beta_2 x_1 + \beta_3 y_1 &= 0 \\ 2\beta_1 + \beta_2 x_2 + \beta_3 y_2 &= 0 \end{aligned} \quad \text{or}$$

$$2\beta_2 x_1 + 2\beta_3 y_1 = \beta_2 x_2 + \beta_3 y_2 \quad \text{or} \quad \beta_3 = \left(\frac{1}{2y_1 - y_2}\right)(x_2 - 2x_1)\beta_2$$

2. Let $a_1 = (1, 2, 3), a_2 = (-1, 2, -1) \in \mathbb{R}^3$.

(a) Is the set a_1, a_2 linearly dependent or linearly independent? Justify.

$\beta_1 a_1 + \beta_2 a_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ means $\beta_1 - \beta_2 = 0 \Rightarrow \beta_1 = \beta_2$. But $2\beta_1 + 2\beta_2 = 0 \Rightarrow \beta_1 = -\beta_2$.

So $\beta_1 = \beta_2 = 0$. Also, you can observe that $a_2 \neq \gamma a_1$ for some $\gamma \in \mathbb{R}$.

(b) If $a_3 = (1, 0, 2)$, is the set a_1, a_2, a_3 linearly dependent or linearly independent? Justify.

$$2a_1 - 2a_2 - 4a_3 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $a_1 - a_2 - 2a_3 = 0$ So a_1, a_2, a_3 are linearly dependent.

Let $a_1 = (1,2,3), a_2 = (-1,2,-1) \in \mathbb{R}^3$.

(c) If $a_4 = (0,0,1)$, is the set a_1, a_2, a_4 linearly dependent or linearly independent? Justify.

If $\beta_1 a_1 + \beta_2 a_2 + \beta_4 a_4 = 0$, then we get the system:

$\beta_1 - \beta_2 = 0$ ① Equations ① and ② imply $\beta_1 = \beta_2 = 0$.

$2\beta_1 + \beta_2 = 0$ ② Now equ. ③ and $\beta_1 = \beta_2 = 0$, means $\beta_3 = 0$

$3\beta_1 - \beta_2 + \beta_4 = 0$ ③ Since $\beta_1 = \beta_2 = \beta_3$ is the only solution, the vectors are linearly independent.

(d) Is the set a_1, a_2, a_3, a_4 linearly dependent or linearly independent? Justify.

They are indendent b/c we already have nonzero β_i's from part ⓑ:

$$1 \cdot a_1 - 1 \cdot a_2 - 2a_3 + 0a_4 = 0.$$

(e) (Spoiler Alert) You should have found that the set a_1, a_2, a_3, a_4 is linearly dependent. Either think up different choices for a_3 and a_4 such that the set a_1, a_2, a_3, a_4 is linearly independent or explain why you think this is not possible.

It is not possible. One way to think about it is

$\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 + \beta_4 a_4 = 0$ will have 3 equations (1 equation for each coordinate) and 4 unknowns $(\beta_1, \beta_2, \beta_3, \beta_4)$. If there is a solution (which there is...) there should be many solutions.

3. Suppose $a_1, a_2, \cdots, a_k$ is a set of n-vectors and 0_n is the n-vector of all zeros. **Demonstrate** that the set $0_n, a_1, a_2, \cdots, a_k$ can never be linearly independent.

Pick c to be any nonzero scalar.

Then $c \cdot 0_n + 0 \cdot a_1 + 0 \cdot a_2 + \dots + 0 \cdot a_n = 0_n$.

not zero

all other scalars are zero.

4. The set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ is linearly dependent and the set $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent. (You should be able to see quickly that this statement is true!)

(a) Show that the vector $w = (1, 7, 4)$ can be written as a linear combination of the vectors in S.

N.t.s $\beta_1, \beta_2, \beta_3$ so that

$$\beta_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix} \quad \text{or}$$

Ans:

$$2 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix}$$

$$\begin{aligned} \beta_1 - \beta_2 &= 1 \\ \beta_1 + \beta_2 + 2\beta_3 &= 7 \\ \beta_1 + \beta_3 &= 4 \end{aligned}$$

solve systematically or guess

$\beta_1 = \beta_3 = 2$

so $\beta_2 = 1$

(b) Show that the vector $w = (1, 7, 4)$ can be written as a linear combination of the vectors in T.

$$\begin{aligned} \beta_1 - \beta_2 &= 1 \Rightarrow \beta_2 = \beta_1 - 1 \\ \beta_1 + \beta_2 + \beta_3 &= 7 \\ \beta_1 + \beta_3 &= 4 \Rightarrow \beta_3 = 4 - \beta_1 \end{aligned}$$

$\beta_1 + \beta_1 - 1 + 4 - \beta_1 = 7$ or

$\beta_1 = 4$, So $\beta_2 = 3$, $\beta_3 = 0$

$$4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix}$$

(c) Are the linear combinations in (a) and (b) above *unique*? That is, for either S or T, is there more than one way to write w in terms of the S or T?

For S, there are multiple ways to write (1,7,4) as a linear combo.

For T, there is only 1 way.