WORKSHEET: INTRODUCTION TO LINEAR INDEPENDENCE

Definitions:

The set of vectors, $a_1, a_2, \cdots, a_k$, is **linearly dependent** if

The set of vectors, $a_1, a_2, \cdots, a_k$, is **linearly independent** if

Ex: $a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Ex: $a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

Equivalent Definitions:

The set of vectors, $a_1, a_2, \cdots, a_k$, is **linearly dependent** if

The set of vectors, $a_1, a_2, \cdots, a_k$, is **linearly independent** if

1. Let $a_1 = (1, 2) \in \mathbb{R}^2$.

   (a) Find at least 3 different choices for a vector $a_2 \in \mathbb{R}^2$ such that the set a_1, a_2 is linearly **dependent**.

   (b) Find a choice for vector $a_2 \in \mathbb{R}^2$ such that a_1, a_2 are linearly **independent** and demonstrate that you are correct or explain why you think this is not possible.

   (c) Find a choices for vector $a_2, a_3 \in \mathbb{R}^2$ such that a_1, a_2, a_3 are linearly **independent** and demonstrate that you are correct or explain why you think this is not possible.

2. Let $a_1 = (1, 2, 3), a_2 = (-1, 2, -1) \in \mathbb{R}^3$.

   (a) Is the set a_1, a_2 linearly dependent or linearly independent? Justify.

   (b) If $a_3 = (1, 0, 2)$, is the set a_1, a_2, a_3 linearly dependent or linearly independent? Justify.

(c) If $a_4 = (0, 0, 1)$, is the set a_1, a_2, a_4 linearly dependent or linearly independent? Justify.

(d) Is the set a_1, a_2, a_3, a_4 linearly dependent or linearly independent? Justify.

(e) (Spoiler Alert) You should have found that the set a_1, a_2, a_3, a_4 is linearly dependent. Either think up different choices for a_3 and a_4 such that the set a_1, a_2, a_3, a_4 is linearly independent or explain why you think this is not possible.

3. Suppose $a_1, a_2, \cdots, a_k$ is a set of n-vectors and 0_n is the n-vector of all zeros. **Demonstrate** that the set $0_n, a_1, a_2, \cdots, a_k$ can never be linearly independent.

4. The set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ is linearly dependent and the set $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent. (You should be able to see quickly that this statement is true!)

   (a) Show that the vector $w = (1, 7, 4)$ can be written as a linear combination of the vectors in S.

   (b) Show that the vector $w = (1, 7, 4)$ can be written as a linear combination of the vectors in T.

   (c) Are the linear combinations in (a) and (b) above *unique*? That is, for either S or T, is there more than one way to write w in terms of the S or T?