1. Definition: A *basis* is a set of n linearly independent n-vectors.

2. A set of n-vectors $a_1, a_2, \cdots, a_k$ is called (mutually) *orthogonal* if a_i is orthogonal to a_j for all $i \neq j$.

   $a_i \perp a_j$ — $a_i^T a_j = 0$

3. Example: $S = \{v_1 = (1,1,1),\ v_2 = (1/2, 1/2, -1),\ v_3 = (1, -1, 0)\}$

   Check: $v_1^T v_2 = \frac{1}{2} + \frac{1}{2} - 1 = 0$ $\quad v_2^T v_3 = \frac{1}{2} - \frac{1}{2} = 0$

   $v_1^T v_3 = 1 - 1 = 0$

4. A vector a is called *normal* if $\|a\| = 1$

   $\frac{1}{4} + \frac{1}{4} + 1 = \frac{6}{4} = \frac{3}{2}$

5. Example:

   $a_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $a_2 = \left(\frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{2\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}\right)$, $a_3 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

6. A set of n-vectors $a_1, a_2, \cdots, a_k$ is called *orthonormal* if the a_i's are mutually orthogonal and normal.

7. Example: $\{a_1, a_2, a_3\}$ orthonormal.

8. Suppose $a_1, a_2, a_3,$ and a_4 is a set of orthonormal n-vectors. Further, suppose that $\beta_1, \beta_2, \beta_3$ and β_4 have the property that

$$\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 + \beta_4 a_4 = 0_n.$$

(a) Take the inner product of a_3 with both sides of the equation above to get a new equation. What can you conclude?

$$a_3^T(\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 + \beta_4 a_4) = a_3^T 0 = 0$$

$$\beta_1 a_3^T a_1 + \beta_2 a_3^T a_2 + \beta_3 a_3^T a_3 + \beta_4 a_3^T a_4 = 0$$

$$0 + 0 + \beta_3 \|a_3\|^2 + 0 = 0$$

$$\beta_3 = 0$$

(b) What can you conclude about β_i for $i = 1, 2, 4$? $\beta_1 = \beta_2 = \beta_4 = 0$ using the same strategy w/ a_1, a_2 and a_4.

(c) What can you conclude about the set $a_1, a_2, a_3,$ and a_4? About *any* set of orthonormal vectors?

They are linearly independent.

! So, a set of n orthonormal n-vectors is a basis

9. Example: Write the vector $x = (1, 2, 3)$ as a linear combination of $T = \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-\sqrt{2}}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$

(a_1, a_2, a_3: 3 orthonormal 3-vectors.)

Want $\beta_1, \beta_2, \beta_3$ so that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 \quad \text{Ugh...}$$

trick Find

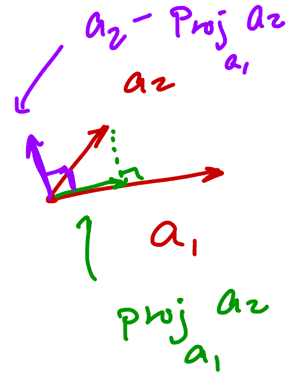
$$x^T a_1 = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3} = \beta_1$$

$$x^T a_2 = \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{-3\sqrt{2}}{\sqrt{3}} = \frac{-\sqrt{3}}{\sqrt{2}} = \beta_2$$

$$x^T a_3 = \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} = \beta_3$$

$$2\sqrt{3}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} - \frac{\sqrt{3}}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -1/2 \\ +1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ +1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



10. Gram-Schmidt Orthogonalization Algorithm
    Strategy: (1) Orthogonalize vectors one-by-one (2) normalize.

start

given: n-vectors $a_1, a_2, \cdots, a_k$

(1) for $i = 1, 2, \cdots, k$

$\text{proj}_{q_{i-1}} a_i$

middle step →
$$\overline{q}_i = a_i - \left(\frac{a_i^T \overline{q}_{i-1}}{\|\overline{q}_{i-1}\|^2}\right)\overline{q}_{i-1} - \left(\frac{a_i^T \overline{q}_{i-2}}{\|\overline{q}_{i-2}\|^2}\right)\overline{q}_{i-2} - \cdots - \left(\frac{a_i^T \overline{q}_1}{\|\overline{q}_1\|^2}\right)\overline{q}_1$$

(2) for $1 = 1, 2, \cdots, k$

$$\text{If } \overline{q}_i \neq 0_n \text{, then } q_i = \left(\frac{1}{\|\overline{q}_i\|}\right)\overline{q}_i.$$

better by $\frac{1}{}$

output: $\{q_i \mid \overline{q}_i \neq 0_n\}$ end.

11. Example: $a_1 = (1, -1, 1),\ a_2 = (1, 0, 1),\ a_3 = (1, 1, 2)$ ← start list

(1) $\overline{q}_2 = a_2 - \left(\frac{a_2^T \overline{q}_1}{\|\overline{q}_1\|^2}\right)\overline{q}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix}$

$\overline{q}_3 = a_3 - \left(\frac{a_3^T \overline{q}_1}{\|q_1\|^2}\right)\overline{q}_1 - \left(\frac{a_3^T \overline{q}_2}{\|q_2\|^2}\right)\overline{q}_2$

$= \begin{bmatrix}1\\1\\2\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} - \left(\frac{5}{2}\right)\begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix} = \begin{bmatrix}-1/2\\0\\1/2\end{bmatrix}$

middle list

$\overline{q}_1 = a_1 = (1, -1, 1)$

$\overline{q}_2 = (1/3, 2/3, 1/3)$

$\overline{q}_3 = (-\frac{1}{2}, 0, \frac{1}{2})$

Ingredients

$a_2^T \overline{q}_1 = 1 + 0 + 1 = 2$

$\|\overline{q}_1\|^2 = 1 + 1 + 1 = 3$

$a_3^T \overline{q}_1 = 1 - 1 + 2 = 2$

$a_3^T \overline{q}_2 = \frac{1}{3} + \frac{2}{3} + \frac{2}{3} = \frac{5}{3}$

$\|\overline{q}_2\|^2 = \frac{1}{9} + \frac{4}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$

$\frac{5/3}{2/3} = \frac{5}{2}$

(2) $q_1 = \overline{q}_1 / \|\overline{q}_1\| = \begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$

$q_2 = \frac{\overline{q}_2}{\|\overline{q}_2\|} = \begin{pmatrix}1/\sqrt{6}\\\sqrt{2}/\sqrt{3}\\1/\sqrt{6}\end{pmatrix}, \quad q_3 = \frac{\overline{q}_3}{\|\overline{q}_3\|} = \begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

output:

q_1, q_2, q_3