WORKSHEET: ORTHONORMAL VECTORS AND GRAM-SCHMIDT ORTHOGONALIZATION

1. Definition: A *basis* is

2. A set of n-vectors $a_1, a_2, \cdots, a_k$ is called *orthogonal* if

3. Example: $S = \{v_1 = (1, 1, 1),\ v_2 = (1/2, 1/2, -1),\ v_3 = (1, -1, 0)\}$

4. A vector a is called *normal* if

5. Example:

6. A set of n-vectors $a_1, a_2, \cdots, a_k$ is called *orthonormal* if

7. Example:

8. Suppose $a_1, a_2, a_3,$ and a_4 is a set of orthonormal n-vectors. Further, suppose that $\beta_1, \beta_2, \beta_3$ and β_4 have the property that

$$\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 + \beta_4 a_4 = 0_n.$$

   (a) Take the inner product of a_3 with both sides of the equation above to get a new equation. What can you conclude?

   (b) What can you conclude about β_i for $i = 1, 2, 4$?

   (c) What can you conclude about the set $a_1, a_2, a_3,$ and a_4? About *any* set of orthonormal vectors?

9. Example: Write the vector $x = (1, 2, 3)$ as a linear combination of $T = \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-\sqrt{2}}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$

10. Gram-Schmidt Orthogonalization Algorithm
    Strategy: (1) Orthogonalize vectors one-by-one (2) normalize.

    given: n-vectors $a_1, a_2, \cdots, a_k$

    (1) for $i = 1, 2, \cdots, k,$ $\qquad \overline{q}_i = a_i - \left(\frac{a_i^T \overline{q}_{i-1}}{\|\overline{q}_{i-1}\|^2}\right) \overline{q}_{i-1} - \left(\frac{a_i^T \overline{q}_{i-2}}{\|\overline{q}_{i-2}\|^2}\right) \overline{q}_{i-2} - \cdots - \left(\frac{a_i^T \overline{q}_1}{\|\overline{q}_1\|^2}\right) \overline{q}_1$

    (2) for $1 = 1, 2, \cdots, k,$ $\qquad$ If $\overline{q}_i \neq 0_n$, then $q_i = \left(\frac{1}{\|\overline{q}_i\|}\right) \overline{q}_i$.

    output: $\{q_i \mid \overline{q}_i \neq 0_n\}$ where q_i's are

    - orthonormal
    - linearly independent and
    - produce the same collection of linear combinations.

    Crucial Point: If the a_i's form a *basis*, then the q_i's form an orthonormal basis.

11. Example: $a_1 = (1, -1, 1),\ a_2 = (1, 0, 1),\ a_3 = (1, 1, 2)$

Here are the steps again in different form:

given: n-vectors $a_1, a_2, \cdots, a_k$

orthogonalize

$\overline{q_1} = a_1$

$\overline{q_2} = a_2 - \left(\frac{a_2^T \overline{q}_1}{\|\overline{q}_1\|^2}\right) \overline{q}_1$

$\overline{q_3} = a_3 - \left(\frac{a_3^T \overline{q}_1}{\|\overline{q}_1\|^2}\right) \overline{q}_1 - \left(\frac{a_3^T \overline{q}_2}{\|\overline{q}_2\|^2}\right) \overline{q}_2$

$\overline{q_4} = a_4 - \left(\frac{a_4^T \overline{q}_1}{\|\overline{q}_1\|^2}\right) \overline{q}_1 - \left(\frac{a_4^T \overline{q}_2}{\|\overline{q}_2\|^2}\right) \overline{q}_2 - \left(\frac{a_4^T \overline{q}_3}{\|\overline{q}_3\|^2}\right) \overline{q}_3$

normalize

$q_1 = \frac{\overline{q}_1}{\|\overline{q}_1\|}$
$q_2 = \frac{\overline{q}_2}{\|\overline{q}_2\|}$
$q_3 = \frac{\overline{q}_3}{\|\overline{q}_3\|}$
$q_4 = \frac{\overline{q}_4}{\|\overline{q}_4\|}$

Note: If a $\overline{q}_i = 0_n$, then toss it out (or) terminate.