$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{bmatrix}, \quad B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 5 \\ 1 & 2 & -1 \end{bmatrix} \end{array}, \quad C = \begin{bmatrix} 3 & 4 \\ 5 & -2 \\ \pi & \sqrt{2} \\ 0 & -7 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad y = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

2×3 columns, rows

3×3

4×2

3×1 vector, column vector

1×3 row vector

reference entries by indices

$B_{23} \equiv$ # in matrix B in row 2 and column 3

$= 5$

1. How to think about a general $m \times n$ matrix A

m rows

n columns

elements A_{ij}

col j

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \text{row } i \rightarrow i \\ \\ m \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & & j & n \end{array} \\ \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & & \\ A_{21} & A_{22} & A_{23} & \cdots & & \\ A_{31} & A_{32} & A_{33} & & & \\ & & & & A_{ij} & \\ & & & & & \\ A_{m1} & & & & & A_{mn} \end{bmatrix}}_{n \text{ col}} \end{array} \left.\vphantom{\begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix}}\right\} m \text{ rows}$$

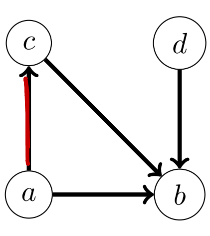
2. Applications

(a) $$\left.\begin{aligned} x_1 + 2x_2 &= 3 \\ 4x_1 - 5x_2 &= 5 \end{aligned}\right] \text{System of linear equations}$$

Info encoded

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \text{cst} \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 5 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{eq 1} \\ \leftarrow \text{eq 2} \end{array}$$

(b) graph example



interpretation

(head, tail)

(c, a) in game, c won

(b, c)

(b, a)

(b, d)

encode

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 5 \\ 1 & 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 \\ 5 & -2 \\ \pi & \sqrt{2} \\ 0 & -7 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad y = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

3. Special Matrices

(a) $\mathbf{0} = \mathbf{0}_{m \times n}$, the zero matrix

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) an $n \times n$ square matrix A and its main diagonal versus its off-diagonal

$$A = \begin{array}{c|c} & 1 \; 2 \cdots n \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \end{array} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

"main diagonal" $\equiv A_{11}, A_{22}, A_{33}, \cdots, A_{nn}$

"off-diagonal" $\equiv A_{ij}$ where $i \neq j$

(c) a diagonal matrix D

square and zeros off-diagonal

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ all zeros

(d) I_n, the $n \times n$ identity matrix is diagonal with 1's on main diagonal

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(e) an upper (lower) triangular matrix A

a square matrix with all zeros below (or above) the main diagonal.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ upper triangular

(f) a block matrix and its submatrices

you can split a matrix into chunks

$$F = \left[\begin{array}{c|c} A & I_2 \\ \hline I_3 & \underset{3\times 2}{0} \end{array}\right] = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 4 & -5 & 6 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & \boxed{6} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 \\ 5 & -2 \\ \pi & \sqrt{2} \\ 0 & -7 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad y = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

4. Things we can do with matrices

(a) transpose of $m \times n$ matrix A, written A^T, is the $n \times m$ matrix obtained by exchanging row + columns of A.

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & -5 \\ 3 & \boxed{6} \end{bmatrix}, \quad y^T = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix},$$

alt: $(A^T)_{ij} = A_{ji}$

$6 = (A^T)_{32} = A_{23}$

(b) matrix addition

of $m \times n$ matrices A and B is

$A + B = G$ where

$G_{ij} = A_{ij} + B_{ij}$

Ex] $A+B$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ 5 & -3 & 9 \end{bmatrix}$$

$A+C$ won't work!

(c) scalar multiplication

α - scalar

A - matrix

αA has entries

αA_{ij}

Ex] $10A = \begin{bmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{bmatrix}$

(d) These operations are well-behaved.

$A + B = B + A$

$\alpha A + \alpha B = \alpha (A+B)$

read book

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 5 \\ 1 & 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 \\ 5 & -2 \\ \pi & \sqrt{2} \\ 0 & -7 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad y = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

(e) **matrix-vector multiplication**

A - $m \times n$ matrix

x - $n \times 1$ col. vector

y - $m \times 1$ col. vector.

we define

$$\underbrace{Ax}_{\text{matrix-scalar mult.}} = y \quad \text{as}$$

$$y_i = A_{i1}x_1 + A_{i2}x_2 + \dots + A_{in}x_n$$

Picture

$$Ax = \begin{array}{c} \\ 1 \\ 2 \\ i^{th}\text{ row} \rightarrow \\ m \end{array} \begin{array}{c} 1 \quad 2 \qquad n \\ \begin{bmatrix} \\ \\ A_{i1} \; A_{i2} \cdots A_{in} \\ \\ \end{bmatrix} \end{array} \begin{array}{c} x \\ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \end{array} = \begin{array}{c} y \\ \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \end{array} \leftarrow y_i$$

$$y_i = \begin{bmatrix} A_{i1} & A_{i2} & \dots & A_{in} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

for every Row in A, we get an entry in y

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{bmatrix} \quad x = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(-1) + 2(1) + 3(2) \\ 4(-1) + (-5)(1) + 6(2) \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$