Monday 4 Nov
- pseudo inverse
- the least squares problem

I. Pseudo inverse (11.5)

A. A is $m \times n$ matrix.

$\boxed{\text{the columns of } A \text{ are linearly independent}}$ if and only if $\boxed{A^T A \text{ is invertible}}$

Why?

Suppose columns of A are linearly independent

Let x be an n-vector so that $A^T A x = 0$. (There must be some vector, e.g. $x=0$.)

So

$$\underset{\text{the number zero}}{0} = \underset{x^T}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T} \underset{n\text{-vector of zeros}}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = x^T (A^T A x) \quad \text{b/c } A^T A x = 0$$

$$= (x^T A^T)(Ax) \quad \text{b/c matrix mult. is associative.}$$

$$= (Ax)^T (Ax) \quad \text{b/c } (A^T B^T) = (BA)^T$$

$$= \|Ax\|^2 \quad \text{b/c } \|a\|^2 = a^T a \ldots$$

But the only way $\|v\|^2 = 0$ is if v is the zero vector.

Since cols of A are linearly independent, the only way $Ax = 0$ is if $x = 0$.

Since the only solution to $A^T A x = 0$ is if $x = 0$, we have shown the columns of $A^T A$ are linearly independent. Thus $A^T A$ is invertible. ✓

Suppose the columns of A are linearly <u>dependent</u>.

Then there is some x so that $Ax=0$ <u>and</u> $x \neq 0$.

$\Rightarrow$ $A^TAx = 0$ and $x \neq 0$.

$\Rightarrow$ columns of A^TA are linearly dependent

$\Rightarrow$ A^TA is not invertible. ✓

<u>def</u>: Suppose A is an $m \times n$ matrix with linearly independent columns. (So $m \geq n$.)
The pseudoinverse of A, denoted A^+, ← dagger
is

$$\underbrace{(A^TA)}^{-1}A^T = A^+.$$

← A^+ is a left inverse of A

We just argued that this makes sense (i.e. A^TA is invertible)

$$\Big(\overset{A^T}{(n\times m)}\overset{A}{(m\times n)}\Big)^{-1}(n\times m) = (n\times n)^{-1}(n\times m) = n\times m$$

See: $A^+A = \left((A^TA)^{-1}A^T\right)A = (A^TA)^{-1}(A^TA) = I_n$

A^+: $(n \times m)$, A: $m \times n$

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix}$.

- Find the pseudo inverse of A.
- Find an inverse by inspection.

So $A^T = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$,

So $A^TA = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$, $(A^TA)^{-1} = \left(\frac{1}{10-1}\right) \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} =$

$= \begin{bmatrix} \frac{2}{9} & \frac{-1}{9} \\ \frac{-1}{9} & \frac{5}{9} \end{bmatrix}$; $(A^TA)^{-1} A^T = \begin{bmatrix} \frac{2}{9} & \frac{-1}{9} \\ \frac{-1}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{4}{9} & \frac{-2}{9} & \frac{-5}{9} \end{bmatrix}$;

check!

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{4}{9} & \frac{-2}{9} & \frac{-5}{9} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{9}+\frac{8}{9} & \frac{1}{9}-\frac{1}{9} \\ \frac{4}{9}-\frac{4}{9} & \frac{4}{9}+\frac{5}{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I. Motivating Question for Least Squares.

$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix}$ with columns $a_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

- The vectors a_1, a_2 do NOT form a basis. Why?
- Find a vector b so that $Ax = b$ has a solution.
- Find a vector c so that $Ax = c$ does not have a solution.

$\boxed{Ex}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, Choose $x_1 = x_2 = 1$.

So $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. So $b = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ would work!

So would choosing $x_1 = 1, x_2 = 0$ so $b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Ex Choose $c = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. There is no solution

to $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, b/c the only

way $c_2 = 2$ is for $x_1 = 1$. Similarly, since $c_3 = -1$,

$x_2 = 1$. But then $c_1 = 2 \neq 5$.

There are many answers: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

The motivating question:

We know $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$ has no solution,

can we find a "good" or "close" solution?
A "best" solution?

Ex $x_1 = x_2 = 1$ gives $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ which is 3 units from $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = c$

but $x_1 = 1, x_2 = 2$ gives $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} = b$

$\|c - b\| = \|(2,0,1)\| = \sqrt{4+1} = \sqrt{5} < 3$. Best? →

Recall $A^{+} = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{4}{9} & \frac{-2}{9} & \frac{-5}{9} \end{bmatrix}$

Find $A^{+} \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{4}{9} & \frac{-2}{9} & \frac{-5}{9} \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5+8-1}{9} \\ \frac{20-4+5}{9} \end{bmatrix}$

$= \begin{bmatrix} \frac{12}{9} \\ \frac{21}{9} \end{bmatrix} = \begin{bmatrix} 4/3 \\ 7/3 \end{bmatrix} = \hat{x} \leftarrow$ best/closest solutin

Check $A\hat{x} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4/3 \\ 7/3 \end{bmatrix} = \begin{bmatrix} 11/3 \\ 8/3 \\ -7/3 \end{bmatrix} = b$

$\| c - b \| = \left\| \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 11/3 \\ 8/3 \\ -7/3 \end{bmatrix} \right\| = \| (\frac{4}{3}, \frac{-2}{3}, \frac{4}{3}) \|$

$= \sqrt{\frac{16+4+16}{9}} = \sqrt{\frac{36}{9}} = \frac{6}{3} = 2$

crazy!