Null Spaces

<u>def</u>: A is an $m \times n$ matrix.

The <u>null space of A</u>, denoted $N(A)$, is the set of all n-vectors, x, such that

$Ax = 0$.

$\boxed{\text{Ex1}}$ $A = \begin{bmatrix} 1 & 2 \\ 10 & 20 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$ is in $N(A)$ and

$y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $N(A)$ because

$Ax = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $Ay = \begin{bmatrix} 3 \\ 30 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\boxed{\text{Ex2}}$ Determine the null space of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}.$$

We need to find all x where $Ax = 0$. To solve:

$$[A \,\vdots\, 0] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 5 & 4 & 9 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \xrightarrow[*]{\text{rref}} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

* Why didn't we solve this using A^{-1}?

$$\begin{array}{ccc} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x_1 \quad + x_3 = 0 \\ x_2 + x_3 = 0 \end{array} \rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = -x_3 \end{array}$$

$$x = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ a \\ -a \end{bmatrix}$$

Answer: $N(A) = \left\{ \begin{bmatrix} a \\ a \\ -a \end{bmatrix} : a \in \mathbb{R} \right\}$

gut check: $\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 10+8-18 \\ 4+8-12 \end{bmatrix} \checkmark$ $\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ \\ \end{bmatrix} \checkmark$

Example 1 Revisited

- Find $N(A)$ for the matrix in Examp6 1.

Answer: $N(A) = \left\{ \begin{bmatrix} 2a \\ -a \end{bmatrix} : a \in \mathbb{R} \right\}$

- Find $\boxed{\text{all}}$ solutions to $Ax = \begin{bmatrix} 3 \\ 30 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 10 & 20 & 30 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 + 2x_2 = 3 \text{ or } x_1 = 3 - 2x_2$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 - 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2a \\ -a \end{bmatrix} : a \in \mathbb{R} \right\}$$

Is $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in this set? (Pick $a = -1$)

vector in $N(A)$

Show ① $N(A) \neq \emptyset$

(1.5) Find a 3×3 matrix A such that
$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

② If x is in $N(A)$, then cx is in $N(A)$ for any number c.

③ If x and y are in $N(A)$, then $x+y$ is in $N(A)$.

④ If x is in $N(A)$ and z is not in $N(A)$, then x and z are linearly independent.

⑤ If x is a solution to $Ax=0$ and z is a solution to $Ax=b$, then $z+x$ is a solution to $Ax=b$.

⑥ If z_1 and z_2 are solutions to $Ax=b$,

then there is an x in $N(A)$ so that $z_2 = z_1 + x$

Big Principles

(1.5) If A is invertible, then $N(A) = \{\vec{0}\}$.

(2+3) $N(A)$ is a subspace.
No linear combination of vectors in $N(A)$ can fall outside of $N(A)$

(5+6) The set of all solutions to $Ax=b$ can be described by finding one particular solution to $Ax=b$, say x_p, and adding it to the vectors in $N(A)$.