Wednesday 23 October

Hybrid Day

- Homework #7 Due today
- Quiz 6 over HW#7 problems on Friday

QR factorization

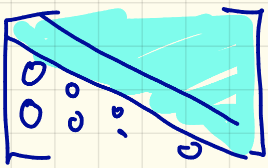
$$\underset{\text{matrix}}{A} = \underbrace{QR}_{\text{product of matrices}}$$

so that

Q is $\boxed{\text{orthogonal}}$ $\begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$ $n \times n$

and

R is $\boxed{\text{upper triangular}}$



where

- $c_i \perp c_j$ and $\|c_i\| = 1$
- c_i's are orthonormal

For class

Mechanics of QR factorization

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = QR$$

we need to show

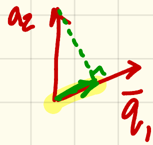
$\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0$ only has 1 solution. Namely $\beta_1 = \beta_2 = \beta_3 = 0$.

① Need column vectors : $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

to be linearly independent. ✓

② Perform Gram-Schmidt on a_1, a_2, a_3 to obtain an orthonomal set of vectors q_1, q_2, q_3.

(2.1) Find $\bar{q}_1, \bar{q}_2, \bar{q}_3$



check $\bar{q}_2 \perp \bar{q}_1$ ✓

$$\bar{q}_1 = a_1$$

$$\bar{q}_2 = a_2 - \boxed{\left(\frac{a_2^T \bar{q}_1}{\|\bar{q}_1\|^2}\right)} \bar{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\bar{q}_3 = a_3 - \left(\frac{a_3^T \bar{q}_2}{\|\bar{q}_2\|^2}\right)\bar{q}_2 - \left(\frac{a_3^T \bar{q}_1}{\|\bar{q}_1\|^2}\right)\bar{q}_1 = \ldots$$

$$\bar{q}_3 = a_3 - \left(\frac{a_3^T \bar{q}_2}{\|\bar{q}_2\|^2}\right)\bar{q}_2 - \left(\frac{a_3^T \bar{q}_1}{\|\bar{q}_1\|^2}\right)\bar{q}_1$$

$1/3$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{\left(-\frac{1}{2}+1\right)}{\frac{6}{4}}\right)\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} - \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/6 \\ -1/6 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

check $\bar{q}_3 \perp \bar{q}_1$ and $\bar{q}_3 \perp \bar{q}_2$ ✓

$-\frac{2}{6} - \frac{2}{6} + \frac{2}{3} = 0$

So, $\bar{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\bar{q}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$, $\bar{q}_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$\div \sqrt{2}$

$\|\bar{q}_1\| = \sqrt{2}$, $\|\bar{q}_2\| = \sqrt{\frac{3}{2}}$, $\|\bar{q}_3\| = \frac{2}{\sqrt{3}}$

Normalize

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} \sqrt{2}/2\sqrt{3} \\ -\sqrt{2}/2\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}, \quad q_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

← check $\|q_1\| = \|q_2\| = \|q_3\| = 1$

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} \sqrt{2}/2\sqrt{3} \\ -\sqrt{2}/2\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}, \quad q_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & \sqrt{2}/2\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & -\sqrt{2}/2\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

Find R

$$R = \begin{bmatrix} a_1^T q_1 & a_2^T q_1 & a_3^T q_1 \\ \boxed{0} & a_2^T q_2 & a_3^T q_2 \\ 0 & 0 & a_3^T q_3 \end{bmatrix}$$

You'd expect this entry to be $a_2^T q_3$

$a_1^T q_3$ $\quad \cancel{a_2^T q_1}$ $\quad \boxed{a_1^T q_2}$

Observation

$a_1^T q_2 = 0$ b/c

$q_2 \perp a_1$ always.

Recall $a_1 = \alpha q_1$

$a_1 = \bar{q}_1 \quad , \quad q_1 = \dfrac{\bar{q}_1}{\|\bar{q}_1\|}$

$$\boxed{\|\bar{q}_1\| q_1 = \bar{q}_1 = a_1}$$

$\boxed{\text{Find } R}$

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} \sqrt{2}/2\sqrt{3} \\ -\sqrt{2}/2\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}, \quad q_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\frac{\sqrt{2}}{2\sqrt{3}} + 0 + \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{2}{2}$$

$$= \frac{3\sqrt{2}}{2\sqrt{3}} = \frac{3}{\sqrt{6}}$$

$$R = \begin{bmatrix} a_1^T q_1 & a_2^T q_1 & a_3^T q_1 \\ \boxed{0} & \boxed{a_2^T q_2} & a_3^T q_2 \\ \boxed{0} & \boxed{0} & a_3^T q_3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \boxed{3/\sqrt{6}} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$$R_{ij} = a_j^T q_i$$

$a_1^T q_3 = 0$ b/c $a_1 \perp q_3$

b/c $q_3 \perp q_1$, $q_1 = \alpha a_1$

$a_2^T q_3 = 0$

b/c $a_2 = \alpha q_1 + \beta q_2$

$\bar{q}_2 = a_2 - \alpha \bar{q}_1$

$a_2 = \bar{q}_2 + \alpha \bar{q}_1$

$$\boxed{a_2 = \beta q_2 + \underline{\gamma} \alpha \underline{q_1}}$$

$a_2^T a_3 = (\beta q_2 + \gamma \alpha q_1)^T a_3$

Why does this work? $A = QR$ always?

① Elementary matrix Algebra

$$\boxed{A = QR} \equiv \boxed{Q^T A = (Q^T Q) R}$$

$$\equiv \boxed{Q^T A = (I_n) R = R}$$

So it is enough to show $\boxed{Q^T A = R}$.

② $Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$

So $Q^T = \begin{bmatrix} — & q_1^T & — \\ — & q_2^T & — \\ — & q_3^T & — \end{bmatrix}$

So

$$Q^T A = \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ - & q_3^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} q_1^T a_1 & q_1^T a_2 & q_1^T a_3 \\ q_2^T a_1 & q_2^T a_2 & \boxed{q_2^T a_3} \\ q_3^T a_1 & q_3^T a_2 & q_3^T a_3 \end{bmatrix}$$

• $a_1 \perp q_2$ and $a_1 \perp q_3$ • $q_3 \perp a_2$

Now, let's **use** the QR-factorization

Motivating Problem

Solve
$$\begin{aligned} x_1 + x_2 &= 2.3 \\ x_1 + x_3 &= 0.18 \\ x_2 + x_3 &= -8.41 \end{aligned}$$

Solve $Ax = b$
or
$(Q^T Q) R x = Q^T b$

$Rx = \underline{\underline{Q^T b}}$

A. Reframe as a matrix equation

S is $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 2.3 \\ 0.18 \\ -8.41 \end{bmatrix}$$

B. Find QR-factorization of A

$$Q = \begin{bmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{6}} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

C. Find $\boxed{Q^T b}$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2.3 \\ 0.18 \\ -8.41 \end{bmatrix} = \begin{bmatrix} 1.6122034 \\ -6.0828995 \\ -5.9640282 \end{bmatrix}$$

Julia ☺

D. Solve $\boxed{Rx = Q^T b}$ via back-substitution

$$\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.6122034 \\ -6.0828995 \\ -5.9640282 \end{bmatrix}$$

OR

$$\sqrt{2}\,x_1 + \frac{1}{\sqrt{2}} x_2 + \frac{1}{\sqrt{2}} x_3 = 1.6122034$$

$$\frac{3}{\sqrt{6}} x_2 + \frac{1}{\sqrt{6}} x_3 = -6.0828995$$

$$\frac{2}{\sqrt{3}} \boxed{x_3} = -5.9640282$$

$$\sqrt{2}x_1 + \frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{2}}x_3 = 1.6122034$$

$$\frac{3}{\sqrt{6}}x_2 + \frac{1}{\sqrt{6}}x_3 = -6.0828995$$

$$\frac{2}{\sqrt{3}}x_3 = -5.9640282$$

So

- $x_3 = (-5.9640282)\frac{\sqrt{3}}{2}$
- $x_2 = \left[(-6.0828995) - \frac{1}{\sqrt{6}}x_3\right]\frac{\sqrt{6}}{3}$
- $x_1 = \left[1.6122034 - \frac{1}{\sqrt{2}}(x_2+x_3)\right] \cdot \frac{1}{\sqrt{2}}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5.445 \\ -3.145 \\ -5.265 \end{bmatrix}$$

Julia :)

QR-backsub

October 22, 2024

[13]:
```
sq2=sqrt(2);
sq6=sqrt(6);
sq3=sqrt(3);
QT=[1/sq2 1/sq2 0;
    1/sq6 -1/sq6 2/sq6;
    -1/sq3 1/sq3 1/sq3];
b=[2.3, 0.18,-8.41];
QTB=QT*b
```

[13]:
```
3-element Vector{Float64}:
  1.7536248173426374
 -6.001249869818787
 -6.07949833456676
```

[14]:
```
x3=QTB[3]*sq3/2
```

[14]:
```
-5.265000000000001
```

[15]:
```
x2=(sq6/3)*(QTB[2] - x3/sq6)
```

[15]:
```
-3.145
```

[16]:
```
x1=(1/sq2)*(QTB[1] - (1/sq2)*(x3+x2))
```

[16]:
```
5.4449999999999985
```

[17]:
```
x=[x1,x2,x3]
```

[17]:
```
3-element Vector{Float64}:
  5.4449999999999985
 -3.145
 -5.265000000000001
```

[18]:
```
A=[1 1 0;
    1 0 1;
    0 1 1];
A*x
```

[18]:

```
3-element Vector{Float64}:
  2.2999999999999985
  0.17999999999999794
 -8.41
```

[19]:

```
##alternatively
A*x-b
```

[19]:

```
3-element Vector{Float64}:
 -1.3322676295501878e-15
 -2.0539125955565396e-15
  0.0
```

[ ]:

```

```

- <u>Summary</u>

with orthonormal columns.

- If Q is an ~~orthonormal~~ matrix;
  then $Q^T Q = I_n$

- Use notation of numbers
  then $\boxed{Q^T = Q^{-1}}$ $\equiv$ the inverse of Q

- $\left(\frac{1}{3}\right)3 = 1$

  $3^{-1} \cdot 3 = 1$

- multiplying by a multiplicative inverse
  $\equiv$
  dividing

- Problems??

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \equiv 0$$

$$\begin{bmatrix} & \\ & \end{bmatrix}$$ no mult. inverse