Mon 21 October

Weather Day!

- Hmwk 7 due Wed
- Quiz 6 on Fri

8.4.a

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} = x$$

"turn upside down"

9×9

$$A x = y$$

9-vector   9-vector

reflection

$$\begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}$$

rotation

$$\begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

QR- Factorization

- $a_1, a_2, \dots, a_n$ $\boxed{m}$-vectors, orthonormal

  This means
  - $\|a_i\| = 1$ for all i
  - for $i \neq j$, $\boxed{a_i \perp a_j} \equiv \boxed{a_i^T a_j = 0}$

- Construct: $A = \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}$

  $= [a_1 \ \ a_2 \ \ \cdots \ \ a_n]$

  an $m \times n$ matrix.

$\cancel{A^2 = AA}$

$\cancel{(m \times n)(m \times n)}$ ~~need $m=n$~~

- $\underset{(n\times m)(m \times n)}{A^T A} = \begin{bmatrix} \text{---} & a_1 & \text{---} \\ \text{---} & a_2 & \text{---} \\ \text{---} & a_3 & \text{---} \\ & \vdots & \\ \text{---} & a_n & \text{---} \end{bmatrix} \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}$

$$
\begin{array}{c}
1
\end{array}
\begin{bmatrix} — a_1^T — \\ — a_2^T — \\ — a_3^T — \\ \\ — a_n^T — \end{bmatrix}
\begin{array}{c}
\quad 2 \\
\begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}
\end{array}
=
\begin{array}{cc}
 & \begin{array}{cccc} & 2 & & \end{array} \\
1 & \begin{bmatrix} 1 & 0 & 0 \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1\ 0 \cdots & 0 \\ 0 & 0 & 0 \cdots 0 & 1 \end{bmatrix}
\end{array}
$$

$$A^T \cdot A = I_n$$

$$(n \times m) \quad (m \times n) \quad (n \times n)$$

the identity matrix

$0 = a_1^T a_2$ b/c $a_1 \perp a_2$

$1 = a_1^T a_1 = \|a_1\|^2 = 1^2$

- <u>Observation</u>: If A has o.n. columns, then $A^T A = I_n$

$$I_n B = B$$

$$5x = 3$$
$$\frac{5x}{5} = \frac{3}{5}$$
$$x = \frac{3}{5}$$

$$(A^T A) x = A^T b$$

$$I_n \cdot x = A^T b$$

$$x = A^T b \checkmark$$

$$\left(\tfrac{1}{5}\right)5x = \left(\tfrac{1}{5}\right)3$$

$$(5^{-1} \cdot 5) x = 5^{-1} \cdot 3$$

$$1 x = 5^{-1} \cdot 3$$

def: $a_1, a_2, \dots, a_n$ n-vector and orthonormal

then $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$ is called

$\boxed{\text{orthogonal.}}$

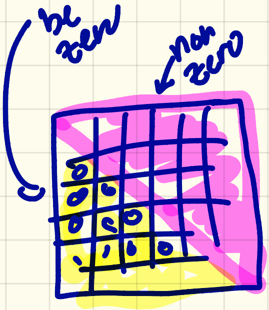
Ex| $\begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix} = I_n$

Ex| orthogonal matrix

$$\begin{array}{c} \begin{bmatrix} \sqrt{2}/2 & -\frac{\sqrt{2}}{2} \\ \sqrt{2}/2 & \frac{\sqrt{2}}{2} \end{bmatrix} \\ \quad a_1 \qquad a_2 \end{array} \quad 2\times 2$$

Sppse A is $n \times n$ matrix and its columns are linearly independent.

Then there exist matrices Q, R so that



① $QR = A$

② Q is orthogonal ($n \times n$) and

③ R is upper triangular

QR factorization

Suppose A is an $n \times n$ matrix with columns $a_1, a_2, \ldots, a_n$ linearly independent.

Then there exist $n \times n$ matrices Q and R such that

- $A = QR$
- Q is orthogonal

  and

- R is upper triangular.

$$\bullet \ \tfrac{1}{2} - \tfrac{1}{6} + \frac{2 \cdot 2}{3 \cdot 2} = \frac{3-1+4}{6} = \frac{6}{6}$$

Ex $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Vectors $a_1 = (1,1,0), a_2 = (1,0,1), a_3 = (0,1,1)$ are linearly independent.

$$Q = \begin{bmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{6}} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \ A$$

normal ✓ orthog ✓ upper Δ

$QR = A$ ✓

$$2x + 3y = 5$$
$$x - 2y = 7$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 7 \end{bmatrix} \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{array}\right]$$

$$x = a_1$$
$$y = a_2$$

$$Ax = b$$

$$\underbrace{\boxed{Q^T Q}}_{I_n} R x = Q^T b$$

$$\boxed{Rx = Q^T b}$$

c_1
c_2
$\vdots$
c_n

back substitution

$$\gamma_{n-2} x_{n-2} + \gamma_{n-1} x_{n-1} + \gamma_n x_n = c_{n-2}$$
$$\beta_{n-1} x_{n-1} + \beta_n x_n = c_{n-1}$$
$$\alpha_n x_n = c_n$$