| Your Name | Your Signature |
|---|---|
| Solutions | |

| Problem | Total Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 12 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 15 | |
| 6 | 15 | |
| 7 | 10 | |
| 8 | 8 | |
| extra credit | 5 | |
| Total | 100 | |

- You have 2 hours.
- If you have a cell phone with you, it should be turned off and put away. (Not in your pocket)
- You may not use a calculator, book, notes or aids of any kind.
- In order to earn partial credit, you must show your work.

1. (20 points)

   (a) State the negation of each statement below.

   i. If $n$ is divisible by 14, then $n$ is divisible by 2 and $n$ is divisible by 7.

   Answer: $n$ is divisible by 14 and ($n$ is not divisible by 2 or $n$ is not divisible by 7.)

   ii. For every real number $r$ there exists a rational number $q$ such that $r < q < r+1$.

   Answer: There exists a real number $r$ such that for every rational number $q$, $q \leq r$ or $r + 1 \geq q$.

   (b) Determine the truth value of the statements below.

   i. $2 \in \mathcal{P}(\{0, 1, 2, 3\})$

   FALSE. Elements of $\mathcal{P}(\{0, 1, 2, 3\})$ are sets.

   ii. $\{\emptyset, \{0, 1\}\} \subseteq \mathcal{P}(\{0, 1, 2, 3\})$

   TRUE. Both $\emptyset$ and $\{0, 1\}$ are elements of $\mathcal{P}(\{0, 1, 2, 3\})$

   (c) List three different partitions of the set $S = \{1, 2, 3\}$. Label your partitions $P_1$, $P_2$, and $P_3$. Use correct notation.

   Some examples: $P_1 = \{\{1, 2, 3\}\}, P_2 = \{\{1, 2\}, \{3\}\}, P_3 = \{\{1\}, \{2, 3\}\}$

   (d) Let $R$ be an equivalence relation on $S = \{a, b, c, d\}$ such that $aRb$ and $dRa$. Circle all of the following statements that *must* also be true.

   i. $cRc$ (TRUE)

   ii. $bRd$ (TRUE)

   iii. $d \in [a]$ (TRUE)

2. (12 points) Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$.

**Proof:** (by induction on $n$.)

Base Step: Let $n = 1$. Observe $1^3 = 1 = \frac{4}{4} = \frac{1^2(1+1)^2}{4}$. Thus, the proposition holds for $n = 1$.

Inductive Step: Suppose $k \in \mathbb{N}$, $n \geq 1$, and $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$. We must show that $1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$. Observe

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \left(\sum_{i=1}^{k} i^3\right) + (k+1)^3 & \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 & \text{by inductive hypothesis} \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} & \text{common denominator} \\
&= \frac{(k+1)^2}{4}\left(k^2 + 4k + 4\right) & \text{factor} \\
&= \frac{(k+1)^2}{4}(k+2)^2,
\end{aligned}$$

which is what we wanted to show.

Thus, by the method of induction, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$.

3. (10 points) Use the method of proof by contrapositive to prove the proposition below.

   Suppose $a, b \in \mathbb{Z}$. If $(a+1)b^2$ is even, then $a$ is odd or $b$ is even.

   **Proof:** We will prove that if $a$ is even and $b$ is odd, then $(a+1)b^2$ is odd.

   Suppose that $a$ is even and $b$ is odd. Thus, there exist integers $m$ and $n$ such that $a = 2m$ and $b = 2n+1$. Thus,

$$(a+1)b^2 = (2m+1)(2n+1)^2 = 2(4mn^2 + 4mn + m + 2n^2 + 2n) + 1,$$

   where $4mn^2 + 4mn + m + 2n^2 + 2n \in \mathbb{Z}$. Thus, $(a+1)b^2$ is odd.

4. (10 points) Use the method of proof by contradiction to proof the proposition below.

   Suppose $a, b \in \mathbb{R}$. If $a$ is rational and $ab$ is irrational, then $b$ is irrational.

   **Proof:** Suppose $a, b \in \mathbb{R}$, $a$ is rational, and $ab$ is irrational. Further, suppose by way of a contradiction that $b$ is rational.

   Since $a$ and $b$ are rational, there exist integers $m, n, p, q$, $n \neq 0$ and $q \neq 0$ such that $a = \frac{m}{n}$ and $b = pq$. Thus, $ab = \frac{mp}{nq}$, where $nq \neq 0$. Thus, $ab$ is rational, which contradicts the assumption that $ab$ is irrational.

   Thus, by the method of proof by contradiction, if $a$ is rational and $ab$ is irrational, then $b$ is irrational.

5. (15 points) Let the function $f : [0, \infty) \to [6, \infty)$ be defined as $f(x) = 3x^2 + 6$. Prove that $f$ is a bijection.

   **Proof:** Let $f : [0, \infty) \to [6, \infty)$ be defined as $f(x) = 3x^2 + 6$.

   (one-to-one) Let $x, x' \in [0, \infty)$ such that $f(x) = f(x')$. Thus, $3x^2 + 6 = 3(x')^2 + 6$. By subtracting 6 and dividing by 3, we obtain the equation $x^2 = (x')^2$. Since both $x$ and $x'$ are nonnegative, $x = x'$. Thus, $f$ is injective.

   (onto) Let $y \in [6, \infty)$. Pick $x = \sqrt{\dfrac{y-6}{3}}$. Observe that since $y \geq 6$, we know $(y-6)/3 \geq 0$. Thus, $x \in [0, \infty)$. Now,

$$f(x) = f\left(\sqrt{\frac{y-6}{3}}\right) = 3\left(\sqrt{\frac{y-6}{3}}\right)^2 + 6 = y.$$

   Thus, $f$ is onto.

   Since $f$ is one-to-one and onto, $f$ is bijective.

6. (15 points) Let $R$ be a relation on $\mathbb{R}$ such that $xRy$ if $x - y \in \mathbb{Z}$.

   (a) Prove that $R$ is an equivalence relation.

   **Proof:** We must show that $R$ is reflexive, symmetric, and transitive.

   (reflexive). Let $x \in \mathbb{R}$. Since $x - x = 0$ and $0 \in \mathbb{Z}$, it follows that $xRx$. Thus, $R$ is reflexive.

   (symmetric). Let $x, y \in \mathbb{R}$ such that $xRy$. By the definition of $R$, it follows that $x - y = n \in \mathbb{Z}$. Thus, $y - x = -n \in \mathbb{Z}$. Thus, $yRx$ and we have shown that $R$ is symmetric.

   (transitive). Let $x, y, z \in \mathbb{R}$ such that $xRy$ and $yRz$. By definition, it follows that $x - y = n \in \mathbb{Z}$ and $y - z = m \in \mathbb{Z}$. Now, $x - z = x - y + y - z = n + m \in \mathbb{Z}$. Thus, $xRz$ and we have shown that $R$ is transitive.

   Since $R$ is reflexive, symmetric and transitive, $R$ is an equivalence relation.

   (b) State three distinct elements in $[\pi]$, the equivalence class of $R$ containing $\pi$.

   There are an infinite number of correct answers. Some include: $-1+\pi, \pi, 1+\pi, 2+\pi, 3+\pi$.

7. (10 points) Let $A, B$, and $C$ be sets. Suppose that $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$. Prove that $A = B$.

**Proof:** Let $A, B$, and $C$ be sets such that $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$.

To show that $A = B$, we must show that $A \subseteq B$ and $B \subseteq A$.

Observe that $A \subseteq B$ by assumption.

(Show $B \subseteq A$.) Let $b \in B$. Since $b \in B$ and $B \subseteq C$, it follows that $b \in C$. Since $b \in C$ and $C \subseteq A$, it follows that $b \in A$. Thus, $B \subseteq A$.

8. (8 points) Demonstrate that the sets $\{0,1\} \times \mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

**Example:** Let $f : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ be defined as: $f(a,n) = \begin{cases} n-1 & \text{if } a = 0 \\ -n & \text{if } a = 1 \end{cases}$.

**5 points extra credit** Prove that your answer above is correct.

We must show that $f$ is a bijection.

(injective) Let $(a,m),(b,n) \in \{0,1\} \times \mathbb{N}$ such that $f(a,m) = f(b,n)$. There are two possibilities: (i) $a = b = 0$ and $m-1 = n-1$ or (ii) $a = b = 1$ and $-m = -n$. Both immediately imply not only that $a = b$ but also $m = n$. Thus, $(a,m) = (b,n)$ and we have shown that the function is injective.

(surjective) Let $n \in \mathbb{N}$. We consider two cases: (i) $n < 0$ and (ii) $n \geq 0$. If $n < 0$, pick element $(1,-n) \in \{0,1\} \times \mathbb{N}$. Now, $f(1,-n) = -(-n) = n$. If $n \geq 0$, pick element $(0,n+1) \in \{0,1\} \times \mathbb{N}$. Then, $f(0,n+1) = (n+1)-1 = n$. Thus, for every $n \in \mathbb{N}$, there exists an $(a,m) \in \{0,1\} \times \mathbb{N}$ such that $f(a,m) = n$. Thus, $f$ is surjective.