1. (15 pts) Give a **direct** proof of the statement below.

   For every real number x, if $x \geq -1$, then there exists some real number y such that $y^2 - 1 = x$.

Proof. Suppose $x \in \mathbb{R}$ such that $x \geq -1$. Choose $y = \sqrt{x+1}$. Observe that since $x \geq -1$, we know $x + 1 \geq 0$ which guarantees that $y = \sqrt{x+1}$ is a real number. Now, we see

$$\begin{aligned} y^2 - 1 &= (\sqrt{x+1})^2 - 1 && \text{by substituting } y = \sqrt{x+1} \\ &= x && \text{by algebra,} \end{aligned}$$

which is what we needed to show. □

2. (15 points) Give a proof by **contrapositive** of the statement below.

   For every pair of integers a and b, if $(a+1)(6a+b)$ is odd, then a is even and b is odd.

Proof. (contrapositive) Let a and b be integers. We will show that if a is odd or b is even, then $(a+1)(6a+b)$ is even.

Suppose that a is odd or b is even. We will proceed by cases.

Case 1: a is odd
Since a is odd, there exists an integer k such that $a = 2k + 1$. Thus,

$$\begin{aligned} (a+1)(6a+b) &= (2k+2)(12k+6+b) && \text{by substituting } a = 2k+1 \\ &= 2(k+1)(12k+6+b) && \text{by factoring} \\ &= 2\ell, \end{aligned}$$

where $\ell = (k+1)(12k+6+b) \in \mathbb{Z}$. Thus, by the definition of even, $(a+1)(6a+b)$ is even.

Case 2: b is even
Since b is even, there exists an integer k such that $b = 2k$. Thus,

$$\begin{aligned} (a+1)(6a+b) &= (a+1)(6a+2k) && \text{by substituting } b = 2k \\ &= 2(a+1)(6a+k) && \text{by factoring} \\ &= 2\ell, \end{aligned}$$

where $\ell = (a+1)(6a+k) \in \mathbb{Z}$. Thus, by the definition of even, $(a+1)(6a+b)$ is even.

Thus, in all cases, $(a+1)(6a+b)$ is even.

□

3. (15 points) Give a proof by **contradiction** of the statement below.

   Let x and y be real numbers. If $5x + 20y = 754$, then either x or y is not an integer.

   Proof. (by contradiction) Suppose x and y are real numbers satisfying $5x + 20y = 754$. For the sake of contradiction we further suppose that both x and y are integers. Upon factoring, we obtain

   $$5(x + 4y) = 754.$$

   Since x and y are integers, it follows that $x + 4y$ is an integer. Thus, $5 \mid 5(x + 4y)$ and, since $5(x + 4y) = 754$, it means $5 \mid 754$. But this contradicts the fact that $5 \nmid 754$. □

4. (20 points) Let A and B be sets. Prove that $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

   Proof. Let A and B be sets.

   ($\Rightarrow$:) Suppose $A \subseteq B$. Let X be an arbitrary element of $\mathcal{P}(A)$. Since $X \in \mathcal{P}(A)$, $X \subseteq A$. Since $X \subseteq A$ and $A \subseteq B$, we know $X \subseteq B$. Since $X \subseteq B$, it follows that $X \in \mathcal{P}(B)$. Thus, every element of $\mathcal{P}(A)$ is en element of $\mathcal{P}(B)$, which implies $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

   ($\Leftarrow$:) Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Let a be an arbitrary element of A. Thus, $\{a\} \subseteq A$ and consquently $\{a\} \in \mathcal{P}(A)$. Since $\{a\} \in \mathcal{P}(A)$ and $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, it follows that $\{a\} \in \mathcal{P}(B)$. Since $\{a\} \in \mathcal{P}(B)$, we know $\{a\} \subseteq B$. Since $\{a\} \subseteq B$, it follows that $a \in B$. Since every element of A is an element of B, we have shown that $A \subseteq B$.

   □

5. (15 points) Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$.

   Proof. Let A, B, and C be sets such that $A \subseteq B$. Let x be an arbitrary element of $C - B$. Thus, $x \in C$ and $x \notin B$. Since $A \subseteq B$, and $x \notin B$, it follows that $x \notin A$. Thus, we know $x \not{i}nA$ and $x \in C$. Thus, $x \in C - A$. Since every element of $C - B$ is an element of $C - A$, we can conclude that $C - B \subseteq C - A$.

   □

6. (20 points) Let a, b, and c be positive integers. Prove that if $c \mid a$ and $c \mid b$, then $c \mid \gcd(a, b)$.

   Proof. Suppose $a, b, c \in \mathbb{Z}$ such that $c \mid a$ and $c \mid b$ and let $d = \gcd(a, b)$. We need to show that $c \mid d$.
   From Proposition 7.1, we know there exist integers k and ℓ such that $d = ka + b\ell$. Since $c \mid a$ and $c \mid b$, we know there are integers m and n such that $cm = a$ and $cn = b$. Now,

   $$\begin{aligned} d &= ka + b\ell \\ &= kcm + bcn \\ &= c(km + bn), \end{aligned}$$

   where $km + bn \in \mathbb{Z}$. Since $c(km + bn) = d$, we know $c \mid d$. □