Your Name

Problem	Total Points	Score
1	20	
2	20	
3	12	
4	12	
5	36	
Total	100	

- You have 1 hour.
- If you have a cell phone with you, it should be turned off and put away. (Not in your pocket)
- You may not use a calculator, book, notes or aids other than a single 3 by 5 notecard.
- In order to lower the writing load, you may use the symbols $\in$, $\exists$, and $\forall$ *if* they are used precisely and literally. You will not be graded on aesthetics, so repeated use of "Thus, ..." will not be penalized.
- You may *not* use the symbol $\Rightarrow$ except parenthetically to indicate the logical structure of an *if-and-only-if* argument.
- You must write in complete sentences with proper punctuation. Every sentence must begin with a capitalized word in English, not a symbol.
- If you are going to use a method *other than a direct proof*, you must state this at the beginning of your proof unless the directions require you to use particular approach.
- When using proof by induction, you *must* indicate when the inductive hypothesis is used.

1. (20 pts)

   (a) (8 pts) **Use the definition** to prove that the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$ defined as $f(x) = \frac{2x}{x-1}$ is injective.

   (b) (8pts) **Use the definition** to prove that the function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $g(m, n) = 5m - 2n$ is surjective.

   (c) (4 pts) Do either f or g have an inverse? Justify your conclusion with a *brief* explanation.

2. (20 pts) Suppose $f : A \to B$ and $g : B \to C$ are both bijections. Prove that $g \circ f : A \to C$ is a bijection.

3. (12 pts) Let T be an arbitrary set of integers. How large must T be in order to guarantee that at least two of them have the same remainder upon division by 7? Prove your answer is best possible.

4. (6 pts each, 12 pts total) For each pair of sets below, show that they have the same cardinality by *finding a bijection from one set to the other.* You do not need to prove your function is a bijection.

   (a) $\mathbb{R}$ and $(-\infty, 100)$

   (b) $\{0, 1, 2\} \times \mathbb{N}$ and $\mathbb{N}$

5. (6 pts each, 36 pts total) Short Answer

   (a) Let $A = \{1, 2, 3, 4\}$. Let R be a relation on $\mathcal{P}(A)$ defined as $X \, R \, Y$ if $X \cup Y = A$. Show that R is not a function.

   (b) Show that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(x, y) = 2^x + y$ is not injective.

(c) Show that the function $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by $g(A) = A \cup \{2\}$ is not surjective.

(d) Let $S = \{(q_1, q_2, q_3) : q_1, q_2, q_3 \in \mathbb{Q}\}$. Is S countable? Give a brief justification.

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x - 2|$. Determine the preimage of the interval $[1, 2]$.

(f) Suppose $f : A \to B$ is a function and $X \subseteq A$. Show that $(f^{-1} \circ f)(X) = X$ may **not** be true for all functions f. (Hint: The notation f^{-1} does *not* mean f has an inverse. It is the name of the inverse **relation** of f.)