1. (a) Prove the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{2\}$ defined by $f(x) = \dfrac{2x}{x-1}$ is injective using the definition.

Proof. Suppose $x_1, x_2 \in \mathbb{R} \setminus \{1\}$ and $f(x_1) = f(x_2)$. Then

$$\frac{2x_1}{x_1 - 1} = \frac{2x_2}{x_2 - 1}.$$

Cross-multiplying gives

$$2x_1(x_2 - 1) = 2x_2(x_1 - 1).$$

Expanding both sides yields

$$2x_1x_2 - 2x_1 = 2x_1x_2 - 2x_2.$$

Subtracting $2x_1x_2$ from both sides gives $-2x_1 = -2x_2$, and therefore $x_1 = x_2$. Since $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function f is injective. ■

(b) Prove the function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $g(m, n) = 5m - 2n$ is surjective using the definition.

Proof. Let $k \in \mathbb{Z}$ be arbitrary. We need to find $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that $5m - 2n = k$. Choose $m = k$ and $n = 2k$. Since $k \in \mathbb{Z}$, we know $m, n \in \mathbb{Z}$ and thus $(k, 2k) \in \mathbb{Z} \times \mathbb{Z}$. Now,

$$g(k, 2k) = 5k - 2(2k) = 5k - 4k = k.$$

Since k was arbitrary, g is surjective. ■

(c) The function f has an inverse, but g does not.

Justification. One can use algebra to construct the inverse of f: $f^{-1}(y) = \dfrac{y}{y-2}$.

The function g is *not* injective. Observe $g(0, 0) = 0 = g(2, 5)$.

2. Prove that if $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

Proof. We must show that $g \circ f$ is both injective and surjective.

Injective.
Suppose $a_1, a_2 \in A$ and $(g \circ f)(a_1) = (g \circ f)(a_2)$ (or equivalently $g(f(a_1)) = g(f(a_2))$).

Since g is injective, it follows that $f(a_1) = f(a_2)$. Since f is injective, it follows that $a_1 = a_2$.

Hence $g \circ f$ is injective.

Surjective.
Let $c \in C$ be arbitrary.

Since g is surjective, there exists $b \in B$ such that $g(b) = c$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$.

Now, observe that $(g \circ f)(a) = g(f(a)) = g(b) = c$.

Since c was arbitrary, $g \circ f$ is surjective.

Since $g \circ f$ is both injective and surjective, it is a bijection. ■

3. Let T be an arbitrary set of integers. How large must T be in order to guarantee that at least two of them have the same remainder upon division by 7? Prove your answer is correct.

   Answer. The set T must have at least $\mathbf{8}$ elements.

   Proof. When an integer is divided by 7, the possible remainders are $0, 1, 2, 3, 4, 5, 6$ — exactly 7 distinct values. Let $S = \{0, 1, 2, 3, 4, 5, 6\}$.
   Let $f : T \to S$ be defined as $f(x)$ is the remainder of x upon division by 7.

   Since $|T| = 8 > 7 = |S|$, by the Pigeonhole Principle, f is not injection. Thus, there must be two element of T, say x and y such that $f(x) = f(y)$. By the definition of f, this implies x and y have the same remainder upon division by 7.

   Observe that 8 is best possible since the set S itself demonstrates that there exist sets of cardinality 7 such that all elements have different remainders. ■

4. (a) We exhibit a bijection $F : \mathbb{R} \to (-\infty, 100)$.

   Define $F(x) = 100 - e^x$.

   Why this works.
   We know that as $x \to \infty$, $F \to -\infty$ and as $x \to -\infty$, $F \to 100$. Moreover, F is continuous and always decreasing.

   (b) We exhibit a bijection $G : \{0, 1, 2\} \times \mathbb{N} \to \mathbb{N}$.

   Define $G(k, n) = 3n + k$

   Why this works.
   It is easy to see that the function is onto since every integer $a \in \mathbb{N}$ can be written in the form $a = 3n+k$, where $k \in \{0, 1, 2\}$. That fact that this form ($a = 3n+k$) is *unique*, thanks to the division algorithm, ensures the function is injective.

5. (a) Show the relation R on $\mathcal{P}(A)$ defined by $X \, R \, Y$ iff $X \cup Y = A$ is not a function.

   Observe that $\{1, 2\}R\{3, 4\}$ and $\{1, 2\}R\{2, 3, 4\}$. Therefore R is not a function. ■

   (b) Show the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(x, y) = 2^x + y$ is not injective.

   Observe $f(1, 3) = 2 + 3 = 5$ and $f(2, 1) = 4 + 1 = 5$.

   (c) Show the function $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by $g(A) = A \cup \{2\}$ is not surjective.

   Observe that by the definition of g every set in the image must contain the element 2. Thus, the set $B = \{1, 3\}$ can never be the image of any set.

(d) Let $S = \{(q_1, q_2, q_3) : q_1, q_2, q_3 \in \mathbb{Q}\}$. Is S contable?

Answer. S is countable. We know this because $S = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$. Since we know $\mathbb{Q}$ is countably infinite and we know the Cartesian product of any finite collection of countably infinite sets is countably infinite, we can conclude that S is countably infinite and thus countable.

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x - 2|$.

Answer. The preimage of $[1, 2]$ is $[0, 1] \cup [3, 4]$.

(f) Show there exists a function $f : A \to B$ and a set $X \subseteq A$ such that $(f^{-1} \circ f)(X) \neq X$.

Let $A = \{1, 2\}$, $B = \{3\}$, and define $f(1) = f(2) = 3$ (the constant function). Let $X = \{1\}$. Then $f(X) = \{3\}$. But, $f^{-1}(f(X)) = f^{-1}(\{3\}) = \{a \in A : f(a) = 3\} = \{1, 2\} \neq \{1\} = X$.